Numerical Methods of Electromagnetic Field Theory I (NFT I) Numerische Methoden der Elektromagnetischen Feldtheorie I (NFT I) /

12th Lecture / 12. Vorlesung

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FIT Discretization of the 3rd and 4th Maxwell's Equation / FIT-Diskretisierung der 3. und 4. Maxwellschen Gleichung

Governing Analytic Equations

Maxwell's equations in integral form /
Maxwellsche Gleichungen in Integralform

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint_S \underline{\mathbf{B}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = -\oint_{C=\partial S}\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dR}} - \iint_S \underline{\mathbf{J}}_{\mathrm{m}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint_S \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = \oint_{C=\partial S}\underline{\mathbf{H}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dR}} - \iint_S \underline{\mathbf{J}}_{\mathrm{e}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}}$$

$$\oiint_{S=\partial V}\underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = \iiint_V \rho_{\mathrm{e}}(\underline{\mathbf{R}},t)\,dV$$

$$\oiint_{S=\partial V}\underline{\mathbf{B}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = \iiint_V \rho_{\mathrm{m}}(\underline{\mathbf{R}},t)\,dV$$

FIT Grid Equations

Maxwell's grid equations /
Maxwellsche Gittergleichungen

$$[\mathbf{S}]\frac{\mathrm{d}}{\mathrm{d}t}\{\mathbf{B}\}(t) = -[\mathbf{curl}][\mathbf{R}]\{\mathbf{E}\}(t) \quad -[\mathbf{S}]\{\mathbf{J}_{\mathrm{m}}\}(t)$$

$$[\widetilde{\boldsymbol{\varepsilon}}][\widetilde{\mathbf{S}}]\frac{\mathrm{d}}{\mathrm{d}t}\{\mathbf{E}\}(t) = [\widetilde{\mathbf{curl}}][\widetilde{\boldsymbol{\nu}}][\widetilde{\mathbf{R}}]\{\mathbf{B}\}(t) - [\widetilde{\mathbf{S}}]\{\mathbf{J}_{\mathrm{e}}\}(t)$$

$$[\widetilde{\mathbf{div}}][\widetilde{\boldsymbol{\varepsilon}}][\widetilde{\mathbf{S}}]\{\mathbf{E}\}(t) = [\widetilde{\mathbf{V}}]\{\boldsymbol{\rho}_{\mathrm{e}}\}(t) = \{\mathbf{Q}_{\mathrm{e}}\}(t)$$

$$[\mathbf{div}][\mathbf{S}]\{\mathbf{B}\}(t) = [\mathbf{V}]\{\boldsymbol{\rho}_{\mathrm{m}}\}(t) = \{\mathbf{Q}_{\mathrm{m}}\}(t)$$

3-D FIT – Electrostatic Case / 3D-FIT – Elektrostatischer Fall

Electric Gauss' grid equation – 3rd Maxwell's grid equation in global matrix form /
Elektrische Gaußsche Gittergleichung – 3. Maxwellsche Gittergleichung in globaler Matrixform

$$\left[\widetilde{\mathbf{div}}\right]\left[\widetilde{\boldsymbol{\varepsilon}}\right]\left[\widetilde{\mathbf{S}}\right]\left\{\mathbf{E}\right\}(t)=\left[\widetilde{\mathbf{V}}\right]\left\{\boldsymbol{\rho}_{\mathrm{e}}\right\}(t)=\left\{\mathbf{Q}_{\mathrm{e}}\right\}(t)$$

$$\frac{\partial}{\partial t}\equiv 0$$

$$\left[\widetilde{\mathbf{div}}\right]\left[\widetilde{\boldsymbol{\varepsilon}}\right]\left[\widetilde{\mathbf{S}}\right]\left\{\mathbf{E}\right\}=\left[\widetilde{\mathbf{V}}\right]\left\{\boldsymbol{\rho}_{\mathrm{e}}\right\}=\left\{\mathbf{Q}_{\mathrm{e}}\right\}$$

$$\underline{\mathbf{E}}(\underline{\mathbf{R}})=-\nabla\Phi_{\mathrm{e}}(\underline{\mathbf{R}})$$

$$\underline{\mathbf{D}}(\underline{\mathbf{R}})=\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{E}}(\underline{\mathbf{R}})$$

$$\begin{aligned}\nabla\bullet\underline{\mathbf{D}}(\underline{\mathbf{R}})&=\rho_{\mathrm{e}}(\underline{\mathbf{R}})\\&=\nabla\bullet\left[\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{E}}(\underline{\mathbf{R}})\right]\\&=\nabla\bullet\left\{\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\bullet\left[-\nabla\Phi_{\mathrm{e}}(\underline{\mathbf{R}})\right]\right\}\\&=-\nabla\bullet\left\{\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\bullet\left[\nabla\Phi_{\mathrm{e}}(\underline{\mathbf{R}})\right]\right\}\end{aligned}$$

Inhomogeneous, anisotropic case /
Inhomogener anisotroper Fall

$$\nabla\bullet\left\{\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\bullet\left[\nabla\Phi_{\mathrm{e}}(\underline{\mathbf{R}})\right]\right\}=-\rho_{\mathrm{e}}(\underline{\mathbf{R}})$$

Homogeneous, isotropic case /
Homogener isotroper Fall

$$\underbrace{\nabla\bullet\nabla}_{=\Delta}\Phi_{\mathrm{e}}(\underline{\mathbf{R}})=-\frac{\rho_{\mathrm{e}}(\underline{\mathbf{R}})}{\varepsilon}$$

$$\Delta\Phi_{\mathrm{e}}(\underline{\mathbf{R}})=-\frac{\rho_{\mathrm{e}}(\underline{\mathbf{R}})}{\varepsilon}$$

3-D FIT – Electrostatic Case / 3D-FIT – Elektrostatischer Fall

$$\underline{\mathbf{E}}(\underline{\mathbf{R}}) = -\nabla\Phi_{\mathrm{e}}(\underline{\mathbf{R}})$$

$$\begin{aligned}\underline{\mathbf{D}}(\underline{\mathbf{R}}) &= \underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}}) \bullet \underline{\mathbf{E}}(\underline{\mathbf{R}}) \\ \nabla \bullet \underline{\mathbf{D}}(\underline{\mathbf{R}}) &= \rho_{\mathrm{e}}(\underline{\mathbf{R}}) \\ &= \nabla \bullet \left[\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}}) \bullet \underline{\mathbf{E}}(\underline{\mathbf{R}})\right] \\ &= \nabla \bullet \left\{\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}}) \bullet \left[-\nabla\Phi_{\mathrm{e}}(\underline{\mathbf{R}})\right]\right\} \\ &= -\nabla \bullet \left\{\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}}) \bullet \left[\nabla\Phi_{\mathrm{e}}(\underline{\mathbf{R}})\right]\right\}\end{aligned}$$

$$\begin{aligned}\underline{\mathbf{D}}(\underline{\mathbf{R}}) &= \underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}}) \bullet \underline{\mathbf{E}}(\underline{\mathbf{R}}) \\ \oiint_{S=\partial V} \underline{\mathbf{D}}(\underline{\mathbf{R}}) \bullet \underline{\mathbf{dS}} &= \iiint_V \rho_{\mathrm{e}}(\underline{\mathbf{R}})\,\mathrm{d}V \\ &= \oiint_{S=\partial V} \left[\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}}) \bullet \underline{\mathbf{E}}(\underline{\mathbf{R}})\right] \bullet \underline{\mathbf{dS}} \\ &= \oiint_{S=\partial V} \left\{\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}}) \bullet \left[-\nabla\Phi_{\mathrm{e}}(\underline{\mathbf{R}})\right]\right\} \bullet \underline{\mathbf{dS}}\nabla \bullet \\ &= -\oiint_{S=\partial V} \left\{\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}}) \bullet \left[\nabla\Phi_{\mathrm{e}}(\underline{\mathbf{R}})\right]\right\} \bullet \underline{\mathbf{dS}}\nabla \bullet\end{aligned}$$

$$-\oiint_{S=\partial V} \left\{\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}}) \bullet \left[\nabla\Phi_{\mathrm{e}}(\underline{\mathbf{R}})\right]\right\} \bullet \underline{\mathbf{dS}} = \iiint_V \rho_{\mathrm{e}}(\underline{\mathbf{R}})\,\mathrm{d}V$$

Inhomogeneous, anisotropic case / Inhomogener anisotroper Fall

$$\nabla \bullet \left\{\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}}) \bullet \left[\nabla\Phi_{\mathrm{e}}(\underline{\mathbf{R}})\right]\right\} = -\rho_{\mathrm{e}}(\underline{\mathbf{R}})$$

Homogeneous, isotropic case / Homogener isotroper Fall

$$\underbrace{\nabla \bullet \nabla}_{=\Delta}\Phi_{\mathrm{e}}(\underline{\mathbf{R}}) = -\frac{\rho_{\mathrm{e}}(\underline{\mathbf{R}})}{\varepsilon}$$

$$\Delta\Phi_{\mathrm{e}}(\underline{\mathbf{R}}) = -\frac{\rho_{\mathrm{e}}(\underline{\mathbf{R}})}{\varepsilon}$$

FIT Discretization of Scalar Electric Potential / FIT-Diskretisierung des skalaren elektrischen Potentials

Differential form / Differentialform

$$\underline{\mathbf{E}}(\underline{\mathbf{R}}) = -\nabla\Phi_\mathrm{e}(\underline{\mathbf{R}})$$

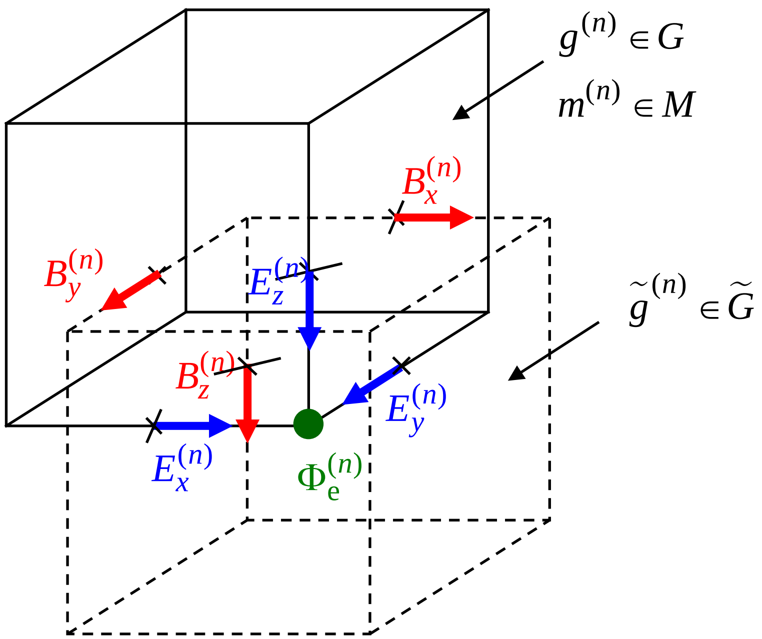
Integral form / Integralform

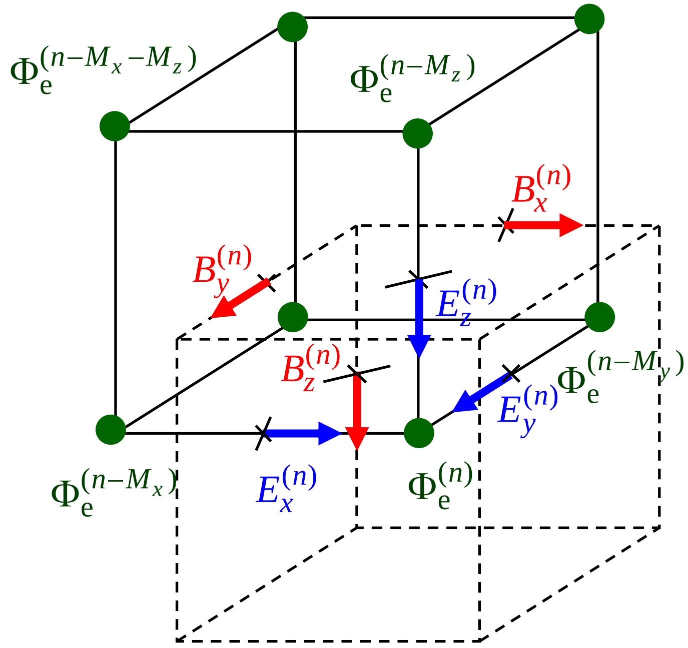
$$\int_C \underline{\mathbf{E}}(\underline{\mathbf{R}}) \bullet \underline{\mathbf{dR}} = -\int_C \nabla\Phi_\mathrm{e}(\underline{\mathbf{R}}) \bullet \underline{\mathbf{dR}}$$
$$= -\left[\Phi_\mathrm{e}(\underline{\mathbf{R}}_2) - \Phi_\mathrm{e}(\underline{\mathbf{R}}_1)\right]$$

FIT grid equation / FIT-Gittergleichung

$$\{E\}^{(n)} = -[R]^{-1}[\mathrm{grad}]\{\Phi_\mathrm{e}\}^{(n)}$$

$$\{\mathbf{E}\} = -[\mathbf{R}]^{-1}[\mathbf{grad}]\{\Phi_\mathrm{e}\}$$





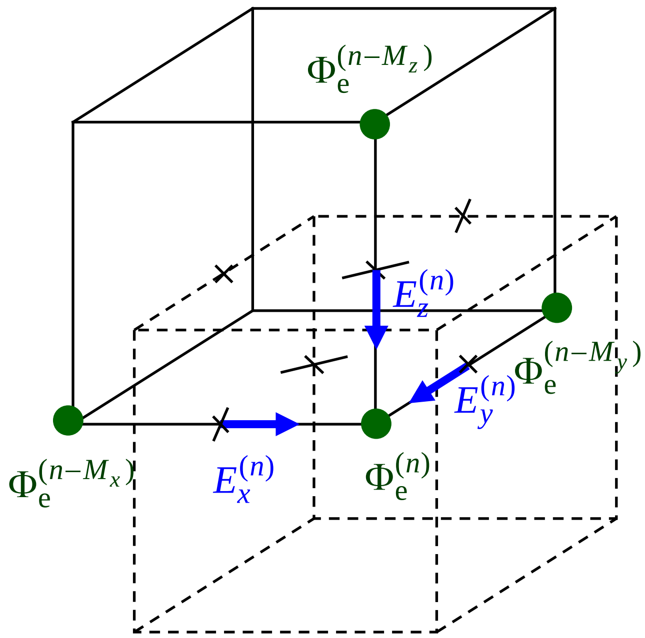
FIT Discretization of Scalar Electric Potential / FIT-Diskretisierung des skalaren elektrischen Potentials

Integral form / Integralform

$$\int_C \underline{\mathbf{E}}(\underline{\mathbf{R}}) \bullet \underline{\mathbf{dR}} = -\int_C \left[\nabla\, \Phi_\mathrm{e}(\underline{\mathbf{R}})\right] \bullet \underline{\mathbf{dR}}$$
$$= -\left[\Phi_\mathrm{e}(\underline{\mathbf{R}}_2) - \Phi_\mathrm{e}(\underline{\mathbf{R}}_1)\right]$$

$$\int_C \underline{\mathbf{E}}(\underline{\mathbf{R}}) \bullet \underline{\mathbf{dR}} = \int_{x=x_0}^{x_0+\Delta x} \underline{\mathbf{E}}(x,y,z) \bullet \underline{\mathbf{dR}}$$
$$= \int_{x=x_0}^{x_0+\Delta x} \underline{\mathbf{E}}(x,y,z) \bullet \underline{\mathrm{e}}_x \, \mathrm{d}x$$
$$= \int_{x=x_0}^{x_0+\Delta x} E_x(x,y,z)\, \mathrm{d}x$$
$$= E_x^{(n)} \int_{x=x_0}^{x_0+\Delta x} \mathrm{d}x$$
$$= E_x^{(n)} \Delta x$$

$$\int_C \left[\nabla\, \Phi_\mathrm{e}(\underline{\mathbf{R}})\right] \bullet \underline{\mathbf{dR}} = \int_{x=x_0}^{x_0+\Delta x} \left[\nabla\, \Phi_\mathrm{e}(x,y,z)\right] \bullet \underline{\mathbf{dR}}$$
$$= \int_{x=x_0}^{x_0+\Delta x} \left[\nabla\, \Phi_\mathrm{e}(x,y,z)\right] \bullet \underline{\mathrm{e}}_x \, \mathrm{d}x$$
$$= \int_{x=x_0}^{x_0+\Delta x} \frac{\partial}{\partial x} \Phi_\mathrm{e}(x,y,z)\, \mathrm{d}x$$
$$= \Phi_\mathrm{e}(x_0,y,z) - \Phi_\mathrm{e}(x_0+\Delta x,y,z)$$
$$= \Phi_\mathrm{e}^{(n-M_x)} - \Phi_\mathrm{e}^{(n)}$$



$$\int_C \underline{\mathbf{E}}(\underline{\mathbf{R}}) \bullet \underline{\mathbf{dR}} = -\int_C \left[\nabla\, \Phi_\mathrm{e}(\underline{\mathbf{R}})\right] \bullet \underline{\mathbf{dR}}$$
$$= -\left[\Phi_\mathrm{e}(\underline{\mathbf{R}}_2) - \Phi_\mathrm{e}(\underline{\mathbf{R}}_1)\right]$$

$$E_x^{(n)} \Delta x = -\Phi_\mathrm{e}^{(n)} - \Phi_\mathrm{e}^{(n-M_x)} = -(I - S_{-M_x})\, \Phi_\mathrm{e}^{(n)}$$
$$E_y^{(n)} \Delta y = -\Phi_\mathrm{e}^{(n)} - \Phi_\mathrm{e}^{(n-M_y)} = -(I - S_{-M_y})\, \Phi_\mathrm{e}^{(n)}$$
$$E_z^{(n)} \Delta z = -\Phi_\mathrm{e}^{(n)} - \Phi_\mathrm{e}^{(n-M_z)} = -(I - S_{-M_z})\, \Phi_\mathrm{e}^{(n)}$$

FIT Discretization of Scalar Electric Potential / FIT-Diskretisierung des skalaren elektrischen Potentials

$$\int_C \underline{\mathbf{E}}(\underline{\mathbf{R}}) \cdot \underline{\mathbf{dR}} = -\int_C \left[\nabla \Phi_\mathrm{e}(\underline{\mathbf{R}})\right] \cdot \underline{\mathbf{dR}} = -\left[\Phi_\mathrm{e}(\underline{\mathbf{R}}_2) - \Phi_\mathrm{e}(\underline{\mathbf{R}}_1)\right]$$

$$\underbrace{\begin{bmatrix} \Delta x & & \\ & \Delta y & \\ & & \Delta z \end{bmatrix}}_{=[R]} \underbrace{\begin{Bmatrix} E_x \\ E_y \\ E_z \end{Bmatrix}^{(n)}}_{=\{E\}^{(n)}} = -\underbrace{\begin{bmatrix} (I - S_{-M_x}) \\ (I - S_{-M_y}) \\ (I - S_{-M_z}) \end{bmatrix}}_{=-[\mathrm{grad}]} \Phi_\mathrm{e}^{(n)}$$

$$E_x^{(n)} \Delta x = -\Phi_\mathrm{e}^{(n)} - \Phi_\mathrm{e}^{(n-M_x)} = -(I - S_{-M_x})\,\Phi_\mathrm{e}^{(n)}$$

$$E_y^{(n)} \Delta y = -\Phi_\mathrm{e}^{(n)} - \Phi_\mathrm{e}^{(n-M_y)} = -(I - S_{-M_y})\,\Phi_\mathrm{e}^{(n)}$$

$$E_z^{(n)} \Delta z = -\Phi_\mathrm{e}^{(n)} - \Phi_\mathrm{e}^{(n-M_z)} = -(I - S_{-M_z})\,\Phi_\mathrm{e}^{(n)}$$

$$[R]\{E\}^{(n)} = -[\mathrm{grad}]\Phi_\mathrm{e}^{(n)}$$

$$[R] = \begin{bmatrix} \Delta x & & \\ & \Delta y & \\ & & \Delta z \end{bmatrix} \rightarrow [R]^{-1} = \begin{bmatrix} \frac{1}{\Delta x} & & \\ & \frac{1}{\Delta y} & \\ & & \frac{1}{\Delta z} \end{bmatrix}$$

$$\{E\}^{(n)} = -[R]^{-1}[\mathrm{grad}]\Phi_\mathrm{e}^{(n)}$$

3-D FIT – Electrostatic Case / 3D-FIT – Elektrostatischer Fall

Electrostatic Poisson's grid equation / Elektrostatische Poissonsche Gittergleichung

$$\underline{\mathbf{E}}(\underline{\mathbf{R}}) = -\nabla \Phi_{\mathrm{e}}(\underline{\mathbf{R}})$$

Inhomogeneous, anisotropic case / Inhomogener anisotroper Fall

$$\nabla \cdot \left\{ \underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}}) \cdot \left[\nabla \Phi_{\mathrm{e}}(\underline{\mathbf{R}}) \right] \right\} = -\rho_{\mathrm{e}}(\underline{\mathbf{R}})$$

Homogeneous, isotropic case / Homogener isotroper Fall

$$\underbrace{\nabla \cdot \nabla}_{=\Delta} \Phi_{\mathrm{e}}(\underline{\mathbf{R}}) = -\frac{\rho_{\mathrm{e}}(\underline{\mathbf{R}})}{\varepsilon}$$

$$\Delta \Phi_{\mathrm{e}}(\underline{\mathbf{R}}) = -\frac{\rho_{\mathrm{e}}(\underline{\mathbf{R}})}{\varepsilon}$$

$$\{\mathbf{E}\} = -[\mathbf{R}]^{-1}[\mathbf{grad}]\{\Phi_{\mathrm{e}}\}$$

$$[\widetilde{\mathbf{div}}][\tilde{\boldsymbol{\varepsilon}}][\tilde{\mathbf{S}}]\{\mathbf{E}\} = -[\widetilde{\mathbf{div}}][\tilde{\boldsymbol{\varepsilon}}][\tilde{\mathbf{S}}][\mathbf{R}]^{-1}[\mathbf{grad}]\{\Phi_{\mathrm{e}}\}$$

$$[\widetilde{\mathbf{div}}][\tilde{\boldsymbol{\varepsilon}}][\tilde{\mathbf{S}}][\mathbf{R}]^{-1}[\mathbf{grad}]\{\Phi_{\mathrm{e}}\} = -[\tilde{\mathbf{V}}]\{\boldsymbol{\rho}_{\mathrm{e}}\}$$

$$[\mathbf{A}]\{\mathbf{x}\} = \{\mathbf{b}\}$$

with / mit

$$[\mathbf{A}] = [\widetilde{\mathbf{div}}][\tilde{\boldsymbol{\varepsilon}}][\tilde{\mathbf{S}}][\mathbf{R}]^{-1}[\mathbf{grad}]$$

$$\{\mathbf{x}\} = \{\Phi_{\mathrm{e}}\}$$

$$\{\mathbf{b}\} = -[\tilde{\mathbf{V}}]\{\boldsymbol{\rho}_{\mathrm{e}}\}$$

3-D FIT – Electrostatic Case / 3D-FIT – Elektrostatischer Fall

Electrostatic Poisson's grid equation / Elektrostatische Poissonsche Gittergleichung

$$\left[\widetilde{\mathbf{div}}\right]\left[\widetilde{\boldsymbol{\varepsilon}}\right]\left[\widetilde{\mathbf{S}}\right]\left[\mathbf{R}\right]^{-1}\left[\mathbf{grad}\right]\left\{\Phi_{\mathrm{e}}\right\} = -\left[\widetilde{\mathbf{V}}\right]\left\{\boldsymbol{\rho}_{\mathrm{e}}\right\}$$

$$\left\{\Phi_{\mathrm{e}}\right\} = \begin{Bmatrix} \Phi_{\mathrm{e}}^{(1)}(t) \\ \Phi_{\mathrm{e}}^{(2)}(t) \\ \vdots \\ \Phi_{\mathrm{e}}^{(N)}(t) \end{Bmatrix}_N \qquad i = x, y, z$$

$$\left[\widetilde{\mathbf{S}}\right] = \begin{bmatrix} \left[\operatorname{diag}\{\Delta y \Delta z\}\right]_{N\times N} & [0] & [0] \\ [0] & \left[\operatorname{diag}\{\Delta x \Delta z\}\right]_{N\times N} & [0] \\ [0] & [0] & \left[\operatorname{diag}\{\Delta x \Delta y\}\right]_{N\times N} \end{bmatrix}_{3N\times 3N}$$

$$\left[\mathbf{R}\right] = \begin{bmatrix} \left[\operatorname{diag}\{\Delta x\}\right]_{N\times N} & [0] & [0] \\ [0] & \left[\operatorname{diag}\{\Delta y\}\right]_{N\times N} & [0] \\ [0] & [0] & \left[\operatorname{diag}\{\Delta z\}\right]_{N\times N} \end{bmatrix}_{3N\times 3N}$$

$$\left[\mathbf{R}\right]^{-1} = \begin{bmatrix} \left[\operatorname{diag}\left\{\frac{1}{\Delta x}\right\}\right]_{N\times N} & [0] & [0] \\ [0] & \left[\operatorname{diag}\left\{\frac{1}{\Delta y}\right\}\right]_{N\times N} & [0] \\ [0] & [0] & \left[\operatorname{diag}\left\{\frac{1}{\Delta z}\right\}\right]_{N\times N} \end{bmatrix}_{3N\times 3N}$$

3-D FIT – Electrostatic Case / 3D-FIT – Elektrostatischer Fall

Electrostatic Poisson's grid equation /
Elektrostatische Poissonsche Gittergleichung

$$[\widetilde{\mathbf{div}}][\widetilde{\boldsymbol{\varepsilon}}][\widetilde{\mathbf{S}}][\mathbf{R}]^{-1}[\mathbf{grad}]\{\Phi_\mathrm{e}\} = -[\widetilde{\mathbf{V}}]\{\boldsymbol{\rho}_\mathrm{e}\}$$

$$[\widetilde{\mathbf{S}}][\mathbf{R}]^{-1} = \begin{bmatrix} [\mathrm{diag}\{\Delta y \Delta z\}]_{N\times N} & [0] & [0] \\ [0] & [\mathrm{diag}\{\Delta x \Delta z\}]_{N\times N} & [0] \\ [0] & [0] & [\mathrm{diag}\{\Delta x \Delta y\}]_{N\times N} \end{bmatrix} \begin{bmatrix} \left[\mathrm{diag}\left\{\dfrac{1}{\Delta x}\right\}\right]_{N\times N} & [0] & [0] \\ [0] & \left[\mathrm{diag}\left\{\dfrac{1}{\Delta y}\right\}\right]_{N\times N} & [0] \\ [0] & [0] & \left[\mathrm{diag}\left\{\dfrac{1}{\Delta z}\right\}\right]_{N\times N} \end{bmatrix}$$

$$= \begin{bmatrix} \left[\mathrm{diag}\left\{\dfrac{\Delta y \Delta z}{\Delta x}\right\}\right]_{N\times N} & [0] & [0] \\ [0] & \left[\mathrm{diag}\left\{\dfrac{\Delta x \Delta z}{\Delta y}\right\}\right]_{N\times N} & [0] \\ [0] & [0] & \left[\mathrm{diag}\left\{\dfrac{\Delta x \Delta y}{\Delta z}\right\}\right]_{N\times N} \end{bmatrix}$$

3-D FIT – Electrostatic Case / 3D-FIT – Elektrostatischer Fall

$$\oiint_{S=\partial V} \underline{\mathbf{D}}(\underline{\mathbf{R}},t) \bullet \underline{\mathbf{dS}} = \iiint_V \rho_\mathrm{e}(\underline{\mathbf{R}},t)\, dV$$

$$\oiint_{S=\partial V} \underline{\mathbf{D}}(\underline{\mathbf{R}},t) \bullet \underline{\mathbf{dS}} = \left(S_{M_x} - I\right) D_x^{(n)}(t)\Delta y \Delta z + \left(S_{M_y} - I\right) D_y^{(n)}(t)\Delta x \Delta z + \left(S_{M_z} - I\right) D_z^{(n)}(t)\Delta x \Delta y$$

$$\oiint_{S=\partial V} \left[\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}}) \bullet \underline{\mathbf{E}}(\underline{\mathbf{R}},t)\right] \bullet \underline{\mathbf{dS}} = \left(S_{M_x} - I\right) \tilde{\varepsilon}_{xx}^{(n)} E_x^{(n)}(t)\Delta y \Delta z + \left(S_{M_y} - I\right) \tilde{\varepsilon}_{yy}^{(n)} E_y^{(n)}(t)\Delta x \Delta z + \left(S_{M_z} - I\right) \tilde{\varepsilon}_{zz}^{(n)} E_z^{(n)}(t)\Delta x \Delta y$$

$$\iiint_V \rho_\mathrm{e}(\underline{\mathbf{R}},t)\, dV = \rho_\mathrm{e}^{(n)}(t)\Delta x \Delta y \Delta z$$

$$E_x^{(n)} = -\frac{1}{\Delta x}\left(\Phi_\mathrm{e}^{(n)} - \Phi_\mathrm{e}^{(n-M_x)}\right) = -\frac{1}{\Delta x}(I - S_{-M_x})\, \Phi_\mathrm{e}^{(n)}$$

$$E_y^{(n)} = -\frac{1}{\Delta y}\left(\Phi_\mathrm{e}^{(n)} - \Phi_\mathrm{e}^{(n-M_y)}\right) = -\frac{1}{\Delta y}(I - S_{-M_y})\, \Phi_\mathrm{e}^{(n)}$$

$$E_z^{(n)} = -\frac{1}{\Delta z}\left(\Phi_\mathrm{e}^{(n)} - \Phi_\mathrm{e}^{(n-M_z)}\right) = -\frac{1}{\Delta z}(I - S_{-M_z})\, \Phi_\mathrm{e}^{(n)}$$

$$\begin{aligned}
\oiint_{S=\partial V} \left[\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}}) \bullet \underline{\mathbf{E}}(\underline{\mathbf{R}},t)\right] \bullet \underline{\mathbf{dS}} &= \left(S_{M_x} - I\right) \tilde{\varepsilon}_{xx}^{(n)} E_x^{(n)}\Delta y \Delta z + \left(S_{M_y} - I\right) \tilde{\varepsilon}_{yy}^{(n)} E_y^{(n)}\Delta x \Delta z + \left(S_{M_z} - I\right) \tilde{\varepsilon}_{zz}^{(n)} E_z^{(n)}\Delta x \Delta y \\
&= \left(S_{M_x} - I\right) \tilde{\varepsilon}_{xx}^{(n)} \left[-\frac{1}{\Delta x}(I - S_{-M_x})\, \Phi_\mathrm{e}^{(n)}\right] \Delta y \Delta z \\
&\quad + \left(S_{M_y} - I\right) \tilde{\varepsilon}_{yy}^{(n)} \left[-\frac{1}{\Delta y}(I - S_{-M_y})\, \Phi_\mathrm{e}^{(n)}\right] \Delta x \Delta z \\
&\quad + \left(S_{M_z} - I\right) \tilde{\varepsilon}_{zz}^{(n)} \left[-\frac{1}{\Delta z}(I - S_{-M_z})\, \Phi_\mathrm{e}^{(n)}\right] \Delta x \Delta y
\end{aligned}$$

3-D FIT – Electrostatic Case / 3D-FIT – Elektrostatischer Fall

$$\iiint_V \rho_e(\underline{\mathbf{R}},t)\,dV = \rho_e^{(n)}(t)\Delta x\Delta y\Delta z$$

$$\begin{aligned}\oiint_{S=\partial V}\left[\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\right]\bullet\underline{\mathbf{dS}} &= \left(S_{M_x}-I\right)\tilde{\varepsilon}_{xx}^{(n)}\left[-\frac{1}{\Delta x}(I-S_{-M_x})\,\Phi_e^{(n)}\right]\Delta y\Delta z\\ &+\left(S_{M_y}-I\right)\tilde{\varepsilon}_{yy}^{(n)}\left[-\frac{1}{\Delta y}(I-S_{-M_y})\,\Phi_e^{(n)}\right]\Delta x\Delta z\\ &+\left(S_{M_z}-I\right)\tilde{\varepsilon}_{zz}^{(n)}\left[-\frac{1}{\Delta z}(I-S_{-M_z})\,\Phi_e^{(n)}\right]\Delta x\Delta y\end{aligned}$$

$$\begin{aligned}&\left(S_{M_x}-I\right)\tilde{\varepsilon}_{xx}^{(n)}\left[-\frac{1}{\Delta x}(I-S_{-M_x})\,\Phi_e^{(n)}\right]\Delta y\Delta z\\ &+\left(S_{M_y}-I\right)\tilde{\varepsilon}_{yy}^{(n)}\left[-\frac{1}{\Delta y}(I-S_{-M_y})\,\Phi_e^{(n)}\right]\Delta x\Delta z\\ &+\left(S_{M_z}-I\right)\tilde{\varepsilon}_{zz}^{(n)}\left[-\frac{1}{\Delta z}(I-S_{-M_z})\,\Phi_e^{(n)}\right]\Delta x\Delta y = \rho_e^{(n)}(t)\Delta x\Delta y\Delta z\end{aligned}$$

$$\begin{aligned}&-\frac{1}{(\Delta x)^2}\left(S_{M_x}-I\right)\tilde{\varepsilon}_{xx}^{(n)}(I-S_{-M_x})\,\Phi_e^{(n)}\\ &-\frac{1}{(\Delta y)^2}\left(S_{M_y}-I\right)\tilde{\varepsilon}_{yy}^{(n)}(I-S_{-M_y})\,\Phi_e^{(n)}\\ &-\frac{1}{(\Delta z)^2}\left(S_{M_z}-I\right)\tilde{\varepsilon}_{zz}^{(n)}(I-S_{-M_z})\,\Phi_e^{(n)} = \rho_e^{(n)}\end{aligned}$$

3-D FIT – Electrostatic Case / 3D-FIT – Elektrostatischer Fall

$$\frac{1}{(\Delta x)^2}\left(S_{M_x}-I\right)\tilde{\varepsilon}_{xx}^{(n)}(I-S_{-M_x})\,\Phi_{\mathrm{e}}^{(n)}=\frac{1}{(\Delta x)^2}\left(S_{M_x}-I\right)(\tilde{\varepsilon}_{xx}^{(n)}-\tilde{\varepsilon}_{xx}^{(n)}S_{-M_x}\Phi_{\mathrm{e}}^{(n)})$$

$$=\frac{1}{(\Delta x)^2}\left[S_{M_x}\tilde{\varepsilon}_{xx}^{(n)}\Phi_{\mathrm{e}}^{(n)}-\underbrace{S_{M_x}\tilde{\varepsilon}_{xx}^{(n)}S_{-M_x}\Phi_{\mathrm{e}}^{(n)}}_{=\tilde{\varepsilon}_{xx}^{(n+M_x)}\underbrace{S_{M_x}S_{-M_x}}_{=I}\Phi_{\mathrm{e}}^{(n)}}-\tilde{\varepsilon}_{xx}^{(n)}\Phi_{\mathrm{e}}^{(n)}+\tilde{\varepsilon}_{xx}^{(n)}S_{-M_x}\Phi_{\mathrm{e}}^{(n)}\right]$$

$$=\frac{1}{(\Delta x)^2}\left[\tilde{\varepsilon}_{xx}^{(n+M_x)}\Phi_{\mathrm{e}}^{(n+M_x)}-\underbrace{\tilde{\varepsilon}_{xx}^{(n+M_x)}\Phi_{\mathrm{e}}^{(n)}-\tilde{\varepsilon}_{xx}^{(n)}\Phi_{\mathrm{e}}^{(n)}}_{=\left[\tilde{\varepsilon}_{xx}^{(n)}+\tilde{\varepsilon}_{xx}^{(n+M_x)}\right]\Phi_{\mathrm{e}}^{(n)}}+\tilde{\varepsilon}_{xx}^{(n)}\Phi_{\mathrm{e}}^{(n-M_x)}\right]$$

$$=\frac{1}{(\Delta x)^2}\left\{\tilde{\varepsilon}_{xx}^{(n+M_x)}S_{M_x}-\left[\underbrace{(I+S_{M_x})\tilde{\varepsilon}_{xx}^{(n)}}_{=2A_{M_x}}\right]I+\tilde{\varepsilon}_{xx}^{(n)}S_{-M_x}\right\}\Phi_{\mathrm{e}}^{(n)}$$

$$=\frac{1}{(\Delta x)^2}\left\{\tilde{\varepsilon}_{xx}^{(n+M_x)}S_{M_x}-\left[2A_{M_x}\tilde{\varepsilon}_{xx}^{(n)}\right]I+\tilde{\varepsilon}_{xx}^{(n)}S_{-M_x}\right\}\Phi_{\mathrm{e}}^{(n)}$$

$$\frac{1}{(\Delta x)^2}\left(S_{M_x}-I\right)\tilde{\varepsilon}_{xx}^{(n)}(I-S_{-M_x})\,\Phi_{\mathrm{e}}^{(n)}=\frac{1}{(\Delta x)^2}\left\{\tilde{\varepsilon}_{xx}^{(n+M_x)}S_{M_x}-\left[2A_{M_x}\tilde{\varepsilon}_{xx}^{(n)}\right]I+\tilde{\varepsilon}_{xx}^{(n)}S_{-M_x}\right\}\Phi_{\mathrm{e}}^{(n)}$$

3-D FIT – Electrostatic Case / 3D-FIT – Elektrostatischer Fall

$$\frac{1}{(\Delta x)^2}\left(S_{M_x}-I\right)\tilde{\varepsilon}_{xx}^{(n)}(I-S_{-M_x})\,\Phi_{\mathrm{e}}^{(n)}=\frac{1}{(\Delta x)^2}\left\{\tilde{\varepsilon}_{xx}^{(n+M_x)}S_{M_x}-\left[2A_{M_x}\tilde{\varepsilon}_{xx}^{(n)}\right]I+\tilde{\varepsilon}_{xx}^{(n)}S_{-M_x}\right\}\Phi_{\mathrm{e}}^{(n)}$$

$$=\left\{\frac{\tilde{\varepsilon}_{xx}^{(n+M_x)}}{(\Delta x)^2}S_{M_x}-\frac{\left[2A_{M_x}\tilde{\varepsilon}_{xx}^{(n)}\right]}{(\Delta x)^2}I+\frac{\tilde{\varepsilon}_{xx}^{(n)}}{(\Delta x)^2}S_{-M_x}\right\}\Phi_{\mathrm{e}}^{(n)}$$

$$\frac{1}{(\Delta y)^2}\left(S_{M_y}-I\right)\tilde{\varepsilon}_{yy}^{(n)}(I-S_{-M_y})\,\Phi_{\mathrm{e}}^{(n)}=\frac{1}{(\Delta y)^2}\left\{\tilde{\varepsilon}_{yy}^{(n+M_y)}\Phi_{\mathrm{e}}^{(n+M_y)}-\left[2S_{M_y}\tilde{\varepsilon}_{yy}^{(n)}\right]\Phi_{\mathrm{e}}^{(n)}+\tilde{\varepsilon}_{yy}^{(n)}\Phi_{\mathrm{e}}^{(n-M_y)}\right\}$$

$$=\left\{\frac{\tilde{\varepsilon}_{yy}^{(n+M_y)}}{(\Delta y)^2}S_{M_y}-\frac{\left[2A_{M_y}\tilde{\varepsilon}_{yy}^{(n)}\right]}{(\Delta y)^2}I+\frac{\tilde{\varepsilon}_{yy}^{(n)}}{(\Delta y)^2}S_{-M_y}\right\}\Phi_{\mathrm{e}}^{(n+)}$$

$$\frac{1}{(\Delta z)^2}\left(S_{M_z}-I\right)\tilde{\varepsilon}_{zz}^{(n)}(I-S_{-M_z})\,\Phi_{\mathrm{e}}^{(n)}=\frac{1}{(\Delta z)^2}\left\{\tilde{\varepsilon}_{zz}^{(n+M_z)}\Phi_{\mathrm{e}}^{(n+M_z)}-\left[(I-S_{M_z})\tilde{\varepsilon}_{zz}^{(n)}\right]\Phi_{\mathrm{e}}^{(n)}+\tilde{\varepsilon}_{zz}^{(n)}\Phi_{\mathrm{e}}^{(n-M_z)}\right\}$$

$$=\left\{\frac{\tilde{\varepsilon}_{zz}^{(n+M_z)}}{(\Delta z)^2}S_{M_z}-\frac{\left[2A_{M_z}\tilde{\varepsilon}_{zz}^{(n)}\right]}{(\Delta z)^2}I+\frac{\tilde{\varepsilon}_{zz}^{(n)}}{(\Delta z)^2}S_{-M_z}\right\}\Phi_{\mathrm{e}}^{(n)}$$

3-D FIT – Electrostatic Case / 3D-FIT – Elektrostatischer Fall

$$\frac{1}{(\Delta x)^2}\left(S_{M_x}-I\right)\tilde{\varepsilon}_{xx}^{(n)}(I-S_{-M_x})\,\Phi_{\mathrm{e}}^{(n)}+\frac{1}{(\Delta z)^2}\left(S_{M_z}-I\right)\tilde{\varepsilon}_{zz}^{(n)}(I-S_{-M_z})\,\Phi_{\mathrm{e}}^{(n)}+\frac{1}{(\Delta y)^2}\left(S_{M_y}-I\right)\tilde{\varepsilon}_{yy}^{(n)}(I-S_{-M_y})\,\Phi_{\mathrm{e}}^{(n)}$$

$$=\left\{\frac{\tilde{\varepsilon}_{xx}^{(n+M_x)}}{(\Delta x)^2}S_{M_x}-\frac{\left[2A_{M_x}\tilde{\varepsilon}_{xx}^{(n)}\right]}{(\Delta x)^2}I+\frac{\tilde{\varepsilon}_{xx}^{(n)}}{(\Delta x)^2}S_{-M_x}\right\}\Phi_{\mathrm{e}}^{(n)}$$

$$+\left\{\frac{\tilde{\varepsilon}_{yy}^{(n+M_y)}}{(\Delta y)^2}S_{M_y}-\frac{\left[2A_{M_y}\tilde{\varepsilon}_{yy}^{(n)}\right]}{(\Delta y)^2}I+\frac{\tilde{\varepsilon}_{yy}^{(n)}}{(\Delta y)^2}S_{-M_y}\right\}\Phi_{\mathrm{e}}^{(n+)}$$

$$+\left\{\frac{\tilde{\varepsilon}_{zz}^{(n+M_z)}}{(\Delta z)^2}S_{M_z}-\frac{\left[2A_{M_z}\tilde{\varepsilon}_{zz}^{(n)}\right]}{(\Delta z)^2}I+\frac{\tilde{\varepsilon}_{zz}^{(n)}}{(\Delta z)^2}S_{-M_z}\right\}\Phi_{\mathrm{e}}^{(n)}$$

$$=\left\{\underbrace{\frac{\tilde{\varepsilon}_{zz}^{(n)}}{(\Delta z)^2}}_{=\alpha_{zz}^{(n)}}S_{-M_z}+\underbrace{\frac{\tilde{\varepsilon}_{yy}^{(n)}}{(\Delta y)^2}}_{=\alpha_{yy}^{(n)}}S_{-M_y}+\underbrace{\frac{\tilde{\varepsilon}_{xx}^{(n)}}{(\Delta x)^2}}_{=\alpha_{xx}^{(n)}}S_{-M_x}-\underbrace{\left\{\frac{\left[2A_{M_x}\tilde{\varepsilon}_{xx}^{(n)}\right]}{(\Delta x)^2}+\frac{\left[2A_{M_y}\tilde{\varepsilon}_{yy}^{(n)}\right]}{(\Delta y)^2}+\frac{\left[2A_{M_z}\tilde{\varepsilon}_{zz}^{(n)}\right]}{(\Delta z)^2}\right\}}_{=\alpha^{(n)}}I+\underbrace{\frac{\tilde{\varepsilon}_{xx}^{(n+M_x)}}{(\Delta x)^2}}_{=\alpha_{xx}^{(n+M_x)}}S_{M_x}+\underbrace{\frac{\tilde{\varepsilon}_{yy}^{(n+M_y)}}{(\Delta y)^2}}_{=\alpha_{yy}^{(n+M_y)}}S_{M_y}+\underbrace{\frac{\tilde{\varepsilon}_{zz}^{(n+M_z)}}{(\Delta z)^2}}_{=\alpha_{zz}^{(n+M_z)}}S_{M_z}\right\}\Phi_{\mathrm{e}}^{(n)}$$

$$=\left\{\alpha_{zz}^{(n)}S_{-M_z}+\alpha_{yy}^{(n)}S_{-M_y}+\alpha_{xx}^{(n)}S_{-M_x}-\alpha^{(n)}I+\alpha_{xx}^{(n+M_x)}S_{M_x}+\alpha_{yy}^{(n+M_y)}S_{M_y}+\alpha_{zz}^{(n+M_z)}S_{M_z}\right\}\Phi_{\mathrm{e}}^{(n)}$$

3-D FIT – Electrostatic Case / 3D-FIT – Elektrostatischer Fall

$$\oint\!\!\oint_{S=\partial V} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = \oint\!\!\oint_{S=\partial V}\left[\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\right]\bullet\underline{\mathbf{dS}}$$

$$= \iiint_V \rho_\mathrm{e}(\underline{\mathbf{R}},t)\,dV$$

$$\oint\!\!\oint_{S=\partial V} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = \oint\!\!\oint_{S=\partial V}\left[\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\right]\bullet\underline{\mathbf{dS}}$$

$$= -\left\{\frac{1}{(\Delta x)^2}\left(S_{M_x}-I\right)\tilde{\varepsilon}_{xx}^{(n)}(I-S_{-M_x})\,\Phi_\mathrm{e}^{(n)} + \frac{1}{(\Delta z)^2}\left(S_{M_z}-I\right)\tilde{\varepsilon}_{zz}^{(n)}(I-S_{-M_z})\,\Phi_\mathrm{e}^{(n)} + \frac{1}{(\Delta y)^2}\left(S_{M_y}-I\right)\tilde{\varepsilon}_{yy}^{(n)}(I-S_{-M_y})\right\}\Phi_\mathrm{e}^{(n)}$$

$$= -\left\{\alpha_{zz}^{(n)}S_{-M_z} + \alpha_{yy}^{(n)}S_{-M_y} + \alpha_{xx}^{(n)}S_{-M_x} - \alpha^{(n)}I + \alpha_{xx}^{(n+M_x)}S_{M_x} + \alpha_{yy}^{(n+M_y)}S_{M_y} + \alpha_{zz}^{(n+M_z)}S_{M_z}\right\}\Phi_\mathrm{e}^{(n)}$$

$$\iiint_V \rho_\mathrm{e}(\underline{\mathbf{R}},t)\,dV = \rho_\mathrm{e}^{(n)}(t)\Delta x\Delta y\Delta z$$

$$-\left(\alpha_{zz}^{(n)}S_{-M_z} + \alpha_{yy}^{(n)}S_{-M_y} + \alpha_{xx}^{(n)}S_{-M_x} - \alpha^{(n)}I + \alpha_{xx}^{(n+M_x)}S_{M_x} + \alpha_{yy}^{(n+M_y)}S_{M_y} + \alpha_{zz}^{(n+M_z)}S_{M_z}\right)\Phi_\mathrm{e}^{(n)} = \rho_\mathrm{e}^{(n)}(t)\Delta x\Delta y\Delta z$$

$$\left(-\alpha_{zz}^{(n)}S_{-M_z} - \alpha_{yy}^{(n)}S_{-M_y} - \alpha_{xx}^{(n)}S_{-M_x} + \alpha^{(n)}I - \alpha_{xx}^{(n+M_x)}S_{M_x} - \alpha_{yy}^{(n+M_y)}S_{M_y} - \alpha_{zz}^{(n+M_z)}S_{M_z}\right)\Phi_\mathrm{e}^{(n)} = \rho_\mathrm{e}^{(n)}(t)\Delta x\Delta y\Delta z$$

$$\left(-\underbrace{\frac{\alpha_{zz}^{(n)}}{\alpha^{(n)}}}_{=\tilde{\alpha}_{zz}^{(n)}}S_{-M_z} - \underbrace{\frac{\alpha_{yy}^{(n)}}{\alpha^{(n)}}}_{=\tilde{\alpha}_{yy}^{(n)}}S_{-M_y} - \underbrace{\frac{\alpha_{xx}^{(n)}}{\alpha^{(n)}}}_{=\tilde{\alpha}_{xx}^{(n)}}S_{-M_x} + I - \underbrace{\frac{\alpha_{xx}^{(n+M_x)}}{\alpha^{(n)}}}_{=\tilde{\alpha}_{xx}^{(n+M_x)}}S_{M_x} - \underbrace{\frac{\alpha_{yy}^{(n+M_y)}}{\alpha^{(n)}}}_{=\tilde{\alpha}_{yy}^{(n+M_y)}}S_{M_y} - \underbrace{\frac{\alpha_{zz}^{(n+M_z)}}{\alpha^{(n)}}}_{=\tilde{\alpha}_{zz}^{(n+M_z)}}S_{M_z}\right)\Phi_\mathrm{e}^{(n)} = \frac{\rho_\mathrm{e}^{(n)}(t)\Delta x\Delta y\Delta z}{\alpha^{(n)}}$$

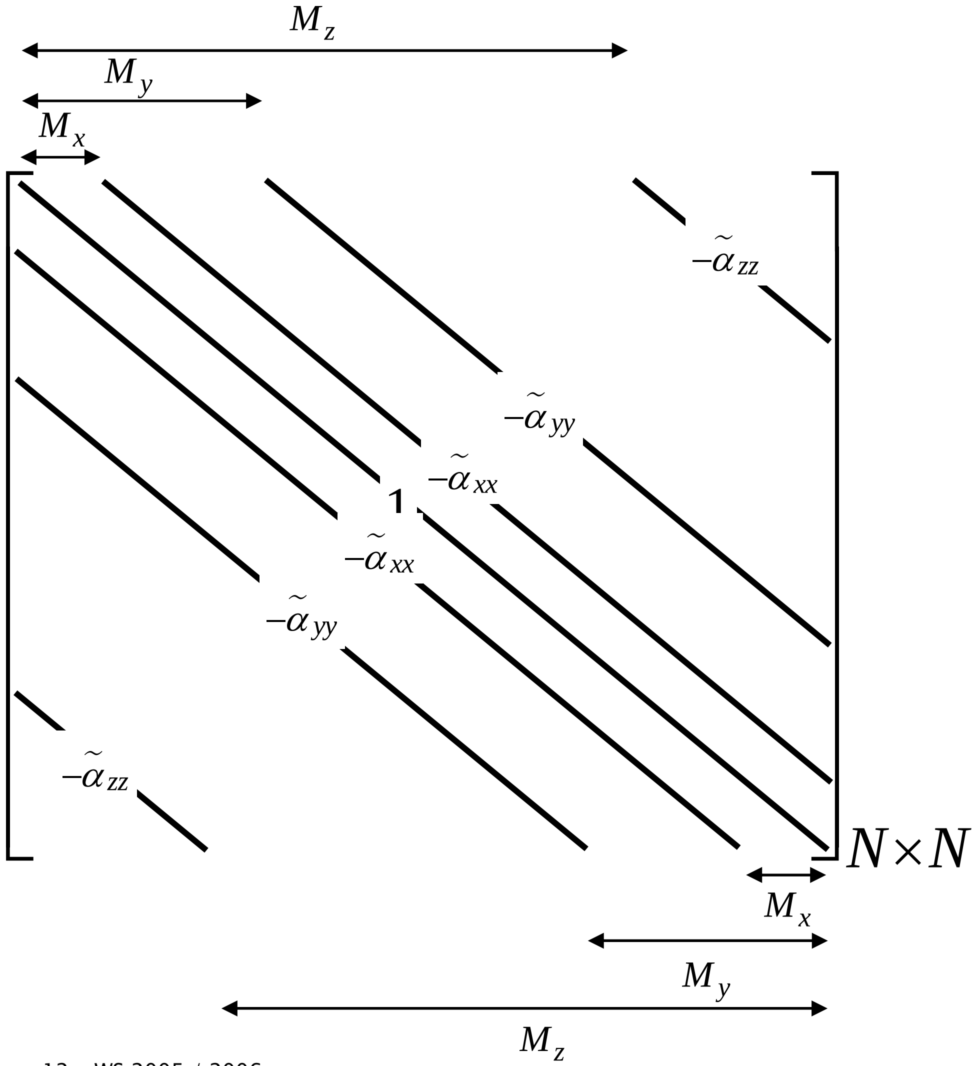
$$\left(-\tilde{\alpha}_{zz}^{(n)}S_{-M_z} - \tilde{\alpha}_{yy}^{(n)}S_{-M_y} - \tilde{\alpha}_{xx}^{(n)}S_{-M_x} + I - \tilde{\alpha}_{xx}^{(n+M_x)}S_{M_x} - \tilde{\alpha}_{yy}^{(n+M_y)}S_{M_y} - \tilde{\alpha}_{zz}^{(n+M_z)}S_{M_z}\right)\Phi_\mathrm{e}^{(n)} = \frac{\rho_\mathrm{e}^{(n)}(t)\Delta x\Delta y\Delta z}{\alpha^{(n)}}$$

Discrete Poisson's Grid Equation / Diskrete Poissonsche Gittergleichung

$$\left(-\tilde{\alpha}_{zz}^{(n)} S_{-M_z} - \tilde{\alpha}_{yy}^{(n)} S_{-M_y} - \tilde{\alpha}_{xx}^{(n)} S_{-M_x} + I - \tilde{\alpha}_{xx}^{(n+M_x)} S_{M_x} - \tilde{\alpha}_{yy}^{(n+M_y)} S_{M_y} - \tilde{\alpha}_{zz}^{(n+M_z)} S_{M_z}\right)\Phi_{\mathrm{e}}^{(n)} = \frac{\rho_{\mathrm{e}}^{(n)}(t)\Delta x \Delta y \Delta z}{\alpha^{(n)}}$$

$$[\mathbf{A}]\{\mathbf{x}\} = \{\mathbf{b}\}$$

$$[\mathbf{A}] =$$



3-D FIT – Electrostatic Case / 3D-FIT – Elektrostatischer Fall

Electrostatic Poisson's grid equation / Elektrostatische Poissonsche Gittergleichung

$$[\widetilde{\mathbf{div}}][\widetilde{\boldsymbol{\varepsilon}}][\widetilde{\mathbf{S}}][\mathbf{R}]^{-1}[\mathbf{grad}]\{\Phi_{\mathrm{e}}\} = -[\widetilde{\mathbf{V}}]\{\boldsymbol{\rho}_{\mathrm{e}}\}$$

Homogeneous isotropic case for a cubic grid complex /
Homogener isotroper Fall für ein kubischen Gitterkomplex

$$[\widetilde{\boldsymbol{\varepsilon}}]_{3N\times 3N} = \varepsilon_0\varepsilon_{\mathrm{r}}[\mathbf{I}]_{3N\times 3N}$$

$$[\widetilde{\mathbf{S}}] = (\Delta x)^2[\mathbf{I}]_{3N\times 3N}$$

$$[\mathbf{R}]^{-1} = \frac{1}{\Delta x}[\mathbf{I}]_{3N\times 3N}$$

$$[\widetilde{\mathbf{V}}] = (\Delta x)^3[\mathbf{I}]_{N\times N}$$

$$[\widetilde{\mathbf{div}}]\varepsilon_0\varepsilon_{\mathrm{r}}[\mathbf{I}](\Delta x)^2[\mathbf{I}]\frac{1}{\Delta x}[\mathbf{I}][\mathbf{grad}]\{\Phi_{\mathrm{e}}\} = -(\Delta x)^3[\mathbf{I}]\{\boldsymbol{\rho}_{\mathrm{e}}\}$$

$$[\widetilde{\mathbf{div}}][\mathbf{grad}]\{\Phi_{\mathrm{e}}\} = -\frac{(\Delta x)^2}{\varepsilon_0\varepsilon_{\mathrm{r}}}\{\boldsymbol{\rho}_{\mathrm{e}}\}$$

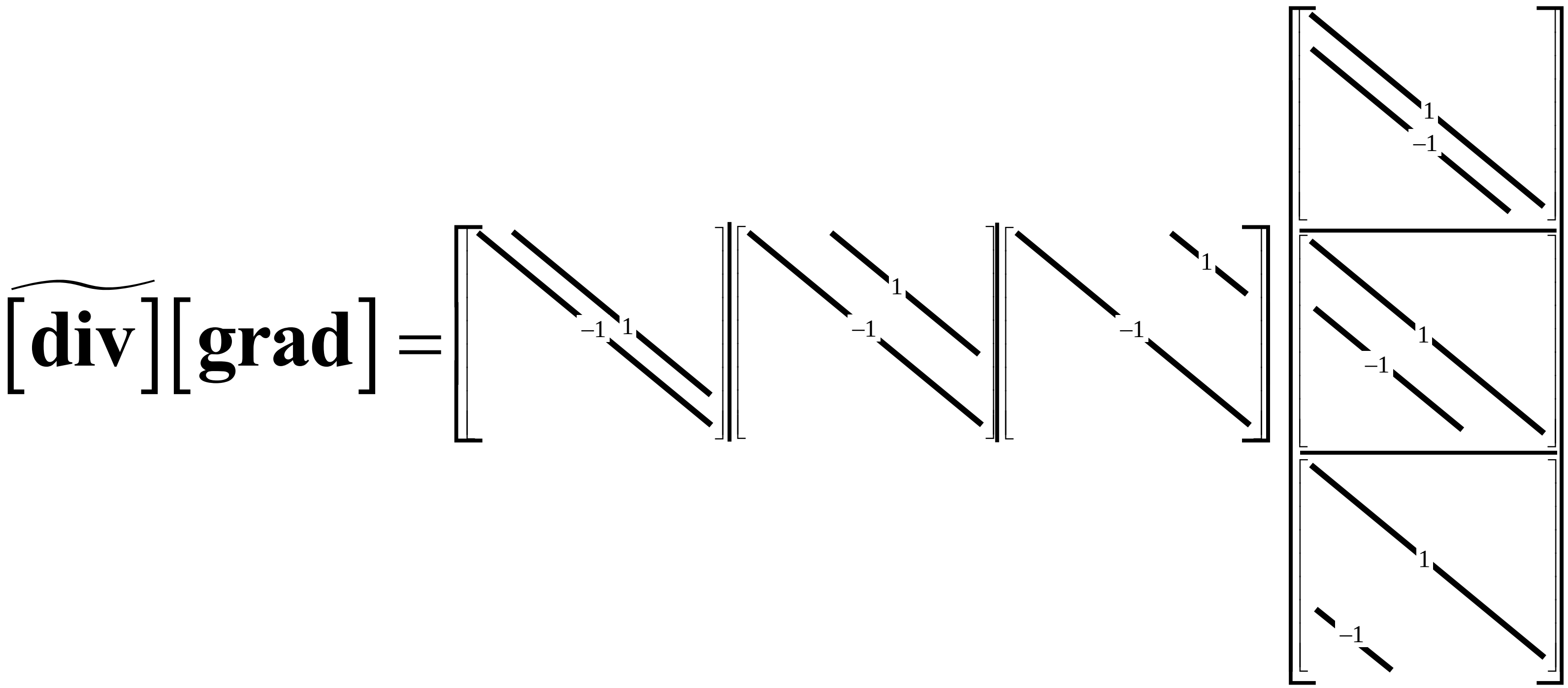
Electrostatic Poisson's grid equation / Elektrostatische Poissonsche Gittergleichung

$$-[\widetilde{\mathbf{div}}][\mathbf{grad}]\{\Phi_{\mathrm{e}}\} = \frac{(\Delta x)^2}{\varepsilon_0\varepsilon_{\mathrm{r}}}\{\boldsymbol{\rho}_{\mathrm{e}}\}$$

Band Structure of the Div-Grad-Operator in Matrix Form / Bandstruktur des Div-Grad-Operators in Matrixform

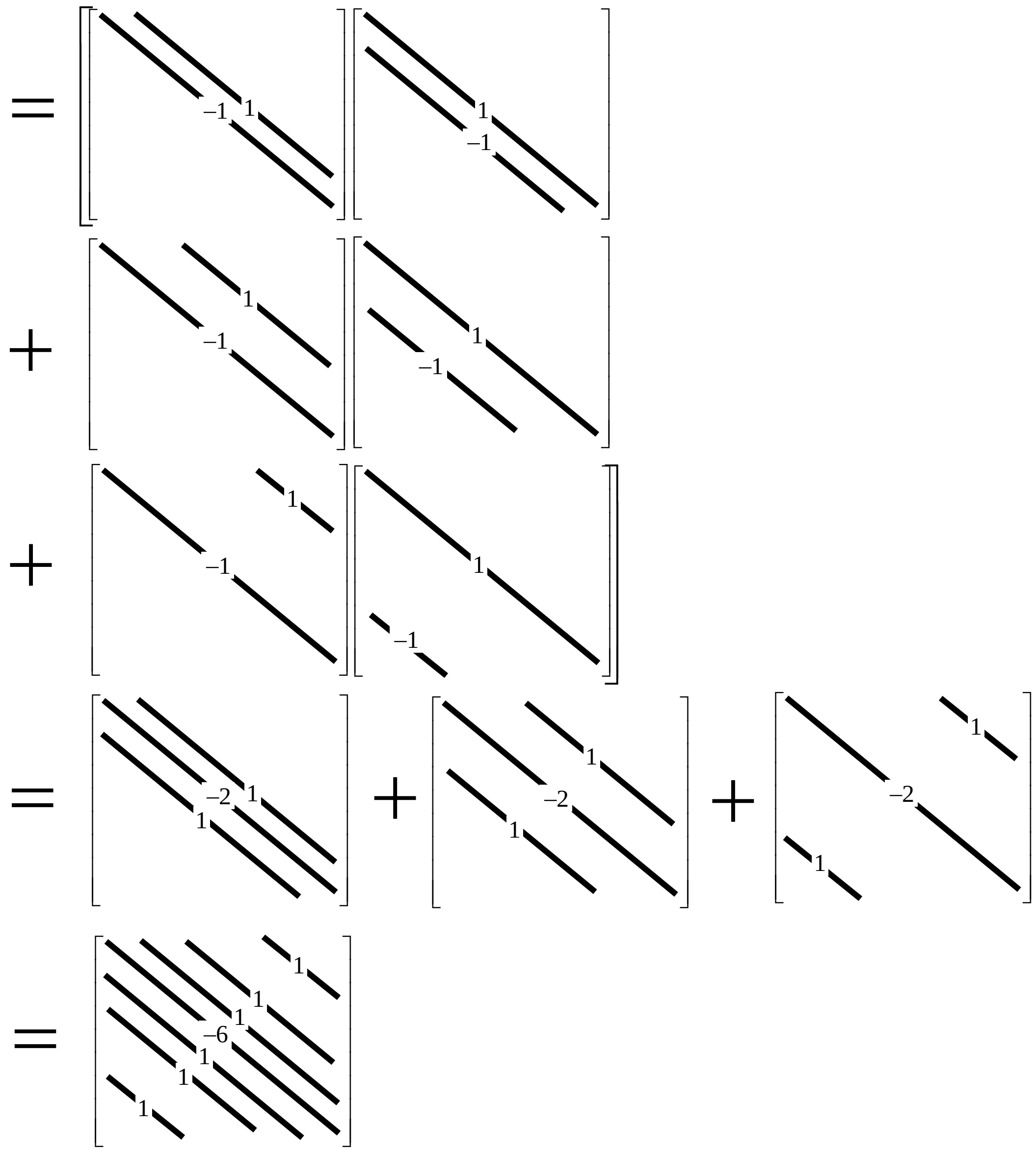
$$[\widetilde{\mathbf{div}}][\mathbf{grad}] = \big[[\mathbf{P}_x],[\mathbf{P}_y],[\mathbf{P}_z]\big] \begin{bmatrix} -[\mathbf{P}_x]^{\mathrm{T}} \\ -[\mathbf{P}_y]^{\mathrm{T}} \\ -[\mathbf{P}_z]^{\mathrm{T}} \end{bmatrix}$$

$$= -\Big[[\mathbf{P}_x][\mathbf{P}_x]^{\mathrm{T}} + [\mathbf{P}_y][\mathbf{P}_y]^{\mathrm{T}} + [\mathbf{P}_z][\mathbf{P}_z]^{\mathrm{T}}\Big]_{N\times N}$$



Band Structure of the Div-Grad-Operator in Matrix Form / Bandstruktur des Div-Grad-Operators in Matrixform

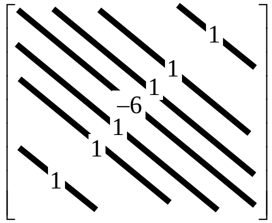
$$[\widetilde{\mathbf{div}}][\mathbf{grad}] =$$



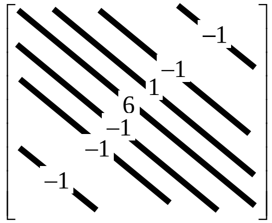
Band Structure of the Div-Grad-Operator in Matrix Form / Bandstruktur des Div-Grad-Operators in Matrixform

3-D case / 3D-Fall

$$\left[\widetilde{\mathbf{div}}\right]\left[\mathbf{grad}\right] =$$

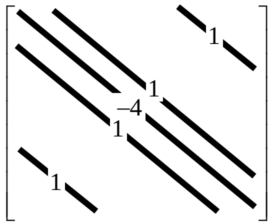


$$-\left[\widetilde{\mathbf{div}}\right]\left[\mathbf{grad}\right] =$$

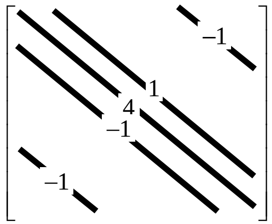


2-D case in the *xz* plane / 2D-Fall in der *xz*-Ebene

$$\left[\widetilde{\mathbf{div}}\right]\left[\mathbf{grad}\right] =$$



$$-\left[\widetilde{\mathbf{div}}\right]\left[\mathbf{grad}\right] =$$



Numerical Methods for Linear Systems / Numerische Methoden für lineare Gleichungssysteme

One the simplest type of iterative methods for solving the linear system /
Einer der einfachsten Typen von iterativen Methoden zur Lösung des linearen Gleichungssystems

$$[\mathbf{A}]\{\mathbf{x}\} = \{\mathbf{b}\}$$

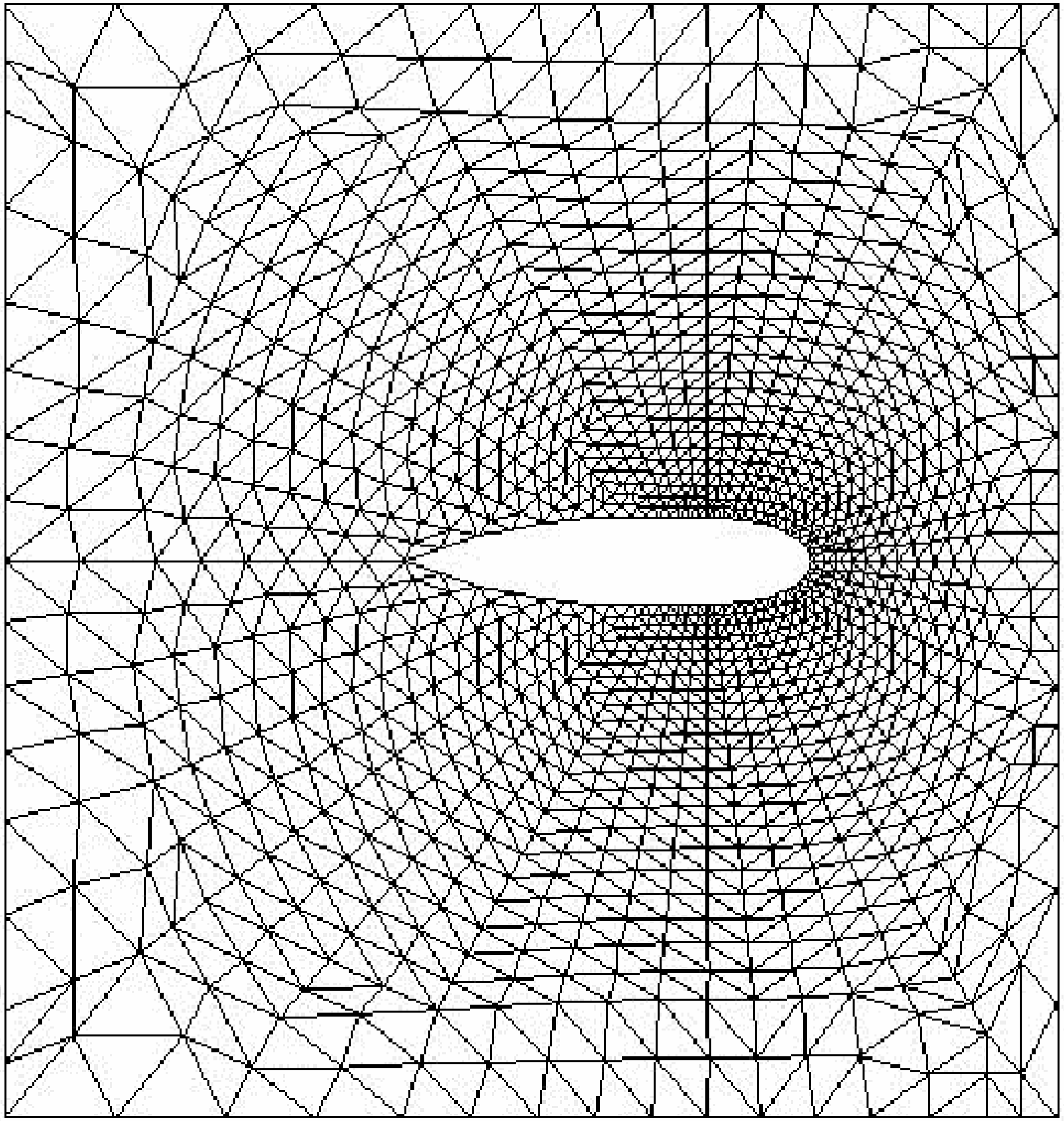
- Gaussian elimination (Gauss method) / Gaußsches Eliminationsverfahren (Gauß-Methode)

- Jacobi method (J method) / Jacobi-Methode (J-Methode)
- Gauss-Seidel method (GS method) / Gauß-Seidel-Methode (GS-Methode)
- Successive overrelaxation method (SOR method) / Überrelaxationsverfahren (SOR-Methode)
- Symmetric successive overrelaxation method (SSOR method) / Symmetrisches Überrelaxationsverfahren (SSOR-Methode)

- Conjugate gradient method (CG method) / Konjugierte Gradientenmethode (KG-Methode)
- Multi grid methods (MG method) / Mehrgitterverfahren (MG-Methode)
- Algebraic multi grid method (AMG method) / Algebraische Mehrgitterverfahren (AMG-Methode)

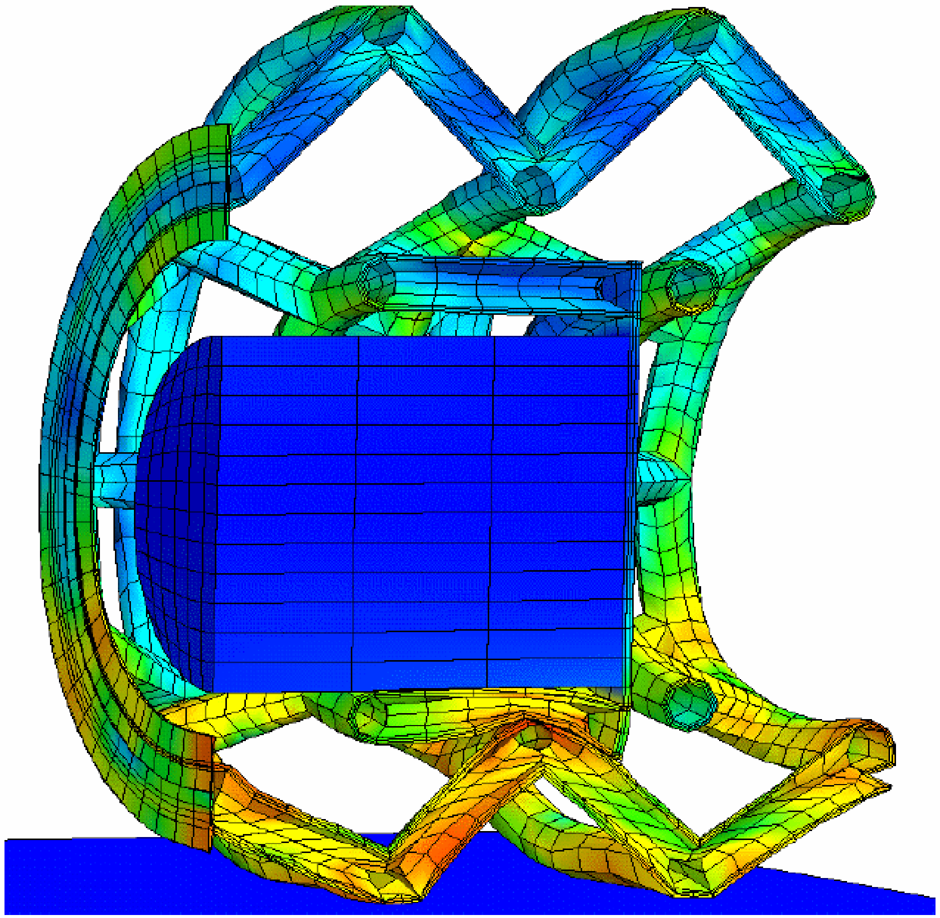
Numerical Methods for Linear Systems / Numerische Methoden für lineare Gleichungssysteme

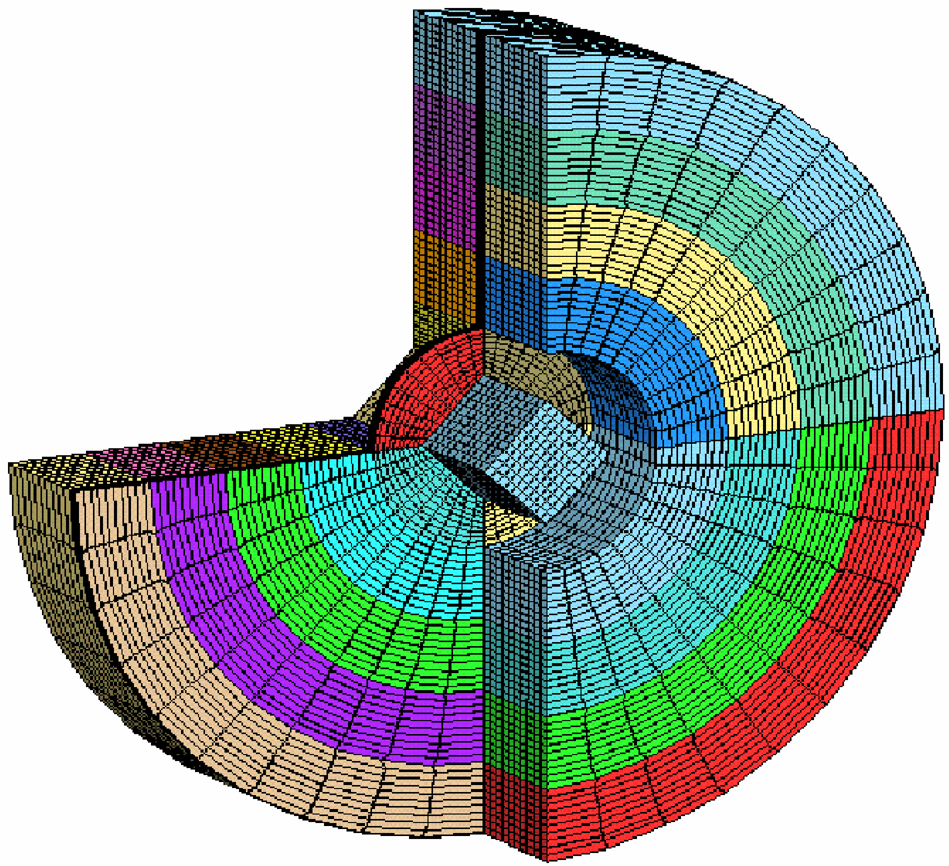
Algebraic multi grid method (AMG method) / Algebraische Mehrgitterverfahren (AMG-Methode)



Numerical Methods for Linear Systems / Numerische Methoden für lineare Gleichungssysteme

Algebraic multi grid method (AMG method) / Algebraische Mehrgitterverfahren (AMG-Methode)





Iterative Methods for Linear Systems / Iterative Methoden für lineare Gleichungssysteme

$$[\mathbf{A}]\{\mathbf{x}\} = \{\mathbf{b}\}$$

LU decomposition of matrix [A] / LU-Zerlegung der Matrix [A]

$$[\mathbf{A}] = [\mathbf{L}] + [\mathbf{D}] + [\mathbf{U}]$$

Lower triangular matrix / Unter Dreiecksmatrix

$$[\mathbf{L}] = \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 \\ A_{21} & 0 & \cdots & & \vdots \\ A_{31} & A_{32} & \ddots & & 0 \\ \vdots & & & 0 & 0 \\ A_{N1} & A_{N2} & \cdots & A_{N(N-1)} & 0 \end{bmatrix}_{N\times N}$$

Main diagonal matrix / Hauptdiagonalmatrix

$$[\mathbf{D}] = \left[\operatorname{diag}\left\{A_{11}, A_{22}, \cdots, A_{NN}\right\}\right]_{N\times N}$$

Upper triangular matrix / Obere Dreiecksmatrix

$$[\mathbf{U}] = \begin{bmatrix} 0 & A_{12} & \cdots & \cdots & A_{1N} \\ 0 & 0 & \cdots & & \vdots \\ 0 & 0 & \ddots & & A_{(N-2)N} \\ \vdots & & & 0 & A_{(N-1)N} \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}_{N\times N}$$

$$[\mathbf{A}]\{\mathbf{x}\} = \{\mathbf{b}\} \quad \rightarrow \quad \{[\mathbf{L}] + [\mathbf{D}] + [\mathbf{U}]\}\{\mathbf{x}\} = \{\mathbf{b}\} \quad \rightarrow \quad [\mathbf{D}]\{\mathbf{x}\} = -\{[\mathbf{L}] + [\mathbf{U}]\}\{\mathbf{x}\} + \{\mathbf{b}\}$$

Iterative Methods for Linear Systems / Iterative Methoden für lineare Gleichungssysteme

$$[\mathbf{A}]\{\mathbf{x}\} = \{\mathbf{b}\}$$

$$\{[\mathbf{L}]+[\mathbf{D}]+[\mathbf{U}]\}\{\mathbf{x}\} = \{\mathbf{b}\}$$

$$[\mathbf{D}]\{\mathbf{x}\} = -\{[\mathbf{L}]+[\mathbf{U}]\}\{\mathbf{x}\} + \{\mathbf{b}\}$$

$$\{\mathbf{x}\} = \underbrace{-[\mathbf{D}]^{-1}\{[\mathbf{L}]+[\mathbf{U}]\}}_{=[\mathbf{G}]}\{\mathbf{x}\} + \underbrace{[\mathbf{D}]^{-1}\{\mathbf{b}\}}_{=\{\mathbf{c}\}}$$

$$= [\mathbf{G}]\{\mathbf{x}\} + \{\mathbf{c}\}$$

$$\{\mathbf{x}\} = [\mathbf{G}]\{\mathbf{x}\} + \{\mathbf{c}\}$$

$$\{\mathbf{x}\}^{(l+1)} = [\mathbf{G}]\{\mathbf{x}\}^{(l)} + \{\mathbf{c}\} \quad l = 1, 2, \ldots, L$$

- Jacobi method (J method) / Jacobi-Methode (J-Methode)
- ⇨ Gauss-Seidel method (GS method) / Gauß-Seidel-Methode (GS-Methode)
- Successive overrelaxation method (SOR method) / Überrelaxationsverfahren (SOR-Methode)
- Symmetric successive overrelaxation method (SSOR method) / Symmetrisches Überrelaxationsverfahren (SSOR-Methode)

Jacobi Method / Jacobi-Methode

$$\{\mathbf{x}\}^{(l+1)} = [\mathbf{G}]_\mathrm{J} \{\mathbf{x}\}^{(l)} + \{\mathbf{c}\}_\mathrm{J} \qquad l = 1, 2, \ldots, L$$

$$[\mathbf{G}]_\mathrm{J} = -[\mathbf{D}]^{-1} \{[\mathbf{L}] + [\mathbf{U}]\}$$

$$\{\mathbf{c}\}_\mathrm{J} = [\mathbf{D}]^{-1} \{\mathbf{b}\}$$

$$x_i^{(l+1)} = \sum_{j=1}^{N} G_{\mathrm{J},ij}\, x_j^{(l)} + c_{\mathrm{J},i} \qquad l = 1, 2, \ldots, L$$

It follows with the LU decomposition of matrix $[\mathbf{A}]$ /
Mit der LU-Zerlegung der Matrix $[\mathbf{A}]$ folgt

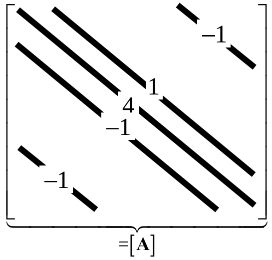
$$x_i^{(l+1)} = \sum_{j=1}^{i-1} G_{\mathrm{J},ij}\, x_j^{(l)} + \sum_{j=i+1}^{N} G_{\mathrm{J},ij}\, x_j^{(l)} + c_{\mathrm{J},i} \qquad l = 1, 2, \ldots, L$$

Jacobi Method / Jacobi-Methode

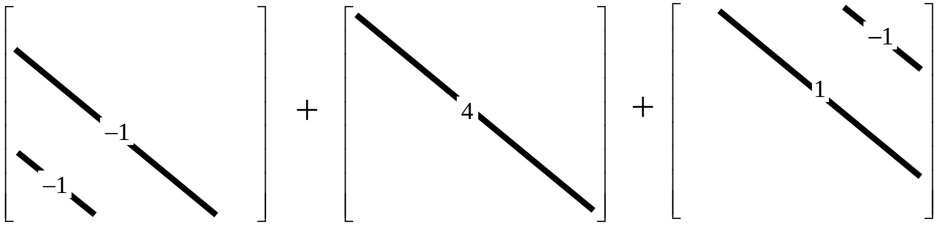
2-D case in the xz plane / 2D-Fall in der xz-Ebene

$$-\widetilde{[\mathbf{div}]}[\mathbf{grad}]\{\Phi_\mathrm{e}\} = \frac{(\Delta x)^2}{\varepsilon_0 \varepsilon_\mathrm{r}} \{\boldsymbol{\rho}_\mathrm{e}\}$$

$$-\widetilde{[\mathbf{div}]}[\mathbf{grad}] =$$

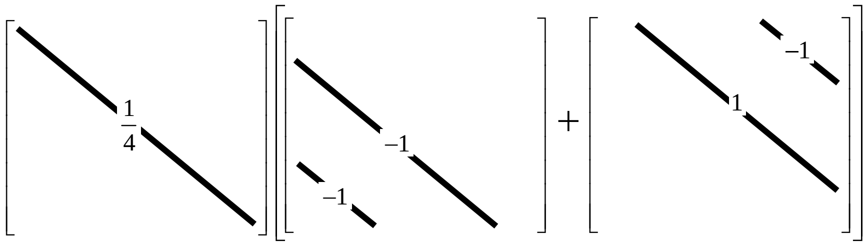


$$[\mathbf{A}] = \{[\mathbf{L}] + [\mathbf{D}] + [\mathbf{U}]\} =$$



$$\{\mathbf{x}\} = \underbrace{-[\mathbf{D}]^{-1}\{[\mathbf{L}]+[\mathbf{U}]\}}_{=[\mathbf{G}]_\mathrm{J}}\{\mathbf{x}\} + \underbrace{[\mathbf{D}]^{-1}\{\mathbf{b}\}}_{=\{\mathbf{c}\}_\mathrm{J}} = [\mathbf{G}]_\mathrm{J}\{\mathbf{x}\} + \{\mathbf{c}\}_\mathrm{J}$$

$$[\mathbf{G}]_\mathrm{J} = -[\mathbf{D}]^{-1}\{[\mathbf{L}]+[\mathbf{U}]\} = -$$



Jacobi & Gauss-Seidel Method / Jacobi- & Gauss-Seidel-Methode

$$\{\mathbf{x}\}^{(l+1)} = [\mathbf{G}]_{\mathrm{J}} \{\mathbf{x}\}^{(l)} + \{\mathbf{c}\}_{\mathrm{J}} \quad l = 1, 2, \ldots, L$$

$$[\mathbf{G}]_{\mathrm{J}} = -[\mathbf{D}]^{-1} \{[\mathbf{L}] + [\mathbf{U}]\}$$

$$\{\mathbf{c}\}_{\mathrm{J}} = [\mathbf{D}]^{-1} \{\mathbf{b}\}$$

$$x^{(n,l+1)} = \frac{1}{4}\left[x^{(n-M_z,l)} + x^{(n-M_x,l)} + x^{(n+M_x,l)} + x^{(n+M_z,l)} + b^{(n)} \right]$$

$$\{\mathbf{x}\}^{(l+1)} = [\mathbf{G}]_{\mathrm{GS}} \{\mathbf{x}\}^{(l)} + \{\mathbf{c}\}_{\mathrm{GS}} \qquad l = 1, 2, \ldots, L$$

$$[\mathbf{G}]_{\mathrm{GS}} = -\left([\mathbf{D}] + [\mathbf{L}]\right)^{-1} [\mathbf{U}]$$

$$\{\mathbf{c}\}_{\mathrm{GS}} = \left([\mathbf{D}] + [\mathbf{L}]\right)^{-1} \{\mathbf{b}\}$$

$$x^{(n,l+1)} = \frac{1}{4}\left[x^{(n-M_z,l+1)} + x^{(n-M_x,l+1)} + x^{(n+M_x,l)} + x^{(n+M_z,l)} + b^{(n)} \right]$$

Successive Overrelaxation Method (SOR Method) / Überrelaxationsverfahren (SOR-Methode)

$$\{\mathbf{x}\}^{(l+1)} = \{\mathbf{x}\}^{(l)} - \omega\left[\overline{\{\mathbf{x}\}}^{(l+1)} - \{\mathbf{x}\}^{(l)}\right] \qquad l = 1, 2, \ldots, L$$

$$= (1-\omega)\{\mathbf{x}\}^{(l)} + \omega\overline{\{\mathbf{x}\}}^{(l+1)}$$

$\{\mathbf{x}\}^{(l+1)}$: Algebraic field vector at the iteration step l / Algebrischer Feldvektor zum Iterationsschritt l

$\overline{\{\mathbf{x}\}}^{(l+1)}$: Gauss-Seidel value at the iteration step l / Gauß-Seidel-Wert zum Iterationsschritt l

ω : Relaxatios Parameter / Relaxationsparameter

$0 < \omega < 1$: Under relaxation method / Unterrelaxationsmethode

$\omega = 1$: Gauss-Seidel method / Gauß-Seidel-Methode

$1 < \omega < 2$: Over relaxation method / Überrelaxationsmethod

Successive Overrelaxation Method (SOR Method) / Überrelaxationsverfahren (SOR-Methode)

$$\{\mathbf{x}\}^{(l+1)} = \{\mathbf{x}\}^{(l)} - \omega\left[\overline{\{\mathbf{x}\}}^{(l+1)} - \{\mathbf{x}\}^{(l)}\right] \qquad l = 1,2,\ldots,L$$

$$= (1-\omega)\{\mathbf{x}\}^{(l)} + \omega\overline{\{\mathbf{x}\}}^{(l+1)}$$

$$x_i^{(l+1)} = (1-\omega)\, x_i^{(l)} + \omega \overline{x}_i^{(l+1)}$$

$$= (1-\omega)\, x_i^{(l)} + \omega\left\{\sum_{j=1}^{i-1} G_{\mathrm{J},ij} \;\; x_j^{(l+1)} + \sum_{j=i+1}^{N} G_{\mathrm{J},ij}\, x_j^{(l)} + c_{\mathrm{J},i}\right\} \qquad i = 1,2,\ldots,N$$

$$x_i^{(l+1)} = (1-\omega)\, x_i^{(l)} + \omega\left\{-\sum_{j=1}^{i-1} D_{ii}^{-1} L_{ij} \;\; x_j^{(l+1)} - \sum_{j=i+1}^{N} D_{ii}^{-1} U_{ij}\, x_j^{(l)} + c_{\mathrm{J},i}\right\} \qquad i = 1,2,\ldots,N$$

$$\{\mathbf{x}\}^{(l+1)} = (1-\omega)\{\mathbf{x}\}^{(l)} - \omega\left\{[\mathbf{D}]^{-1}[\mathbf{L}]\{\mathbf{x}\}^{(l+1)} + [\mathbf{D}]^{-1}[\mathbf{U}]\{\mathbf{x}\}^{(l)} - [\mathbf{D}]^{-1}\{\mathbf{b}\}\right\} \qquad i = 1,2,\ldots,N$$

$$\{\mathbf{x}\}^{(l+1)} = \underbrace{\left([\mathbf{D}] + \omega[\mathbf{L}]\right)^{-1}\left[(1-\omega)[\mathbf{D}] - \omega[\mathbf{U}]\right]}_{=[\mathbf{G}]_{\mathrm{SOR}}}\{\mathbf{x}\}^{(l)} + \underbrace{\omega\left([\mathbf{D}] + \omega[\mathbf{L}]\right)^{-1}\{\mathbf{b}\}}_{=\{\mathbf{c}\}_{\mathrm{SOR}}} \qquad i = 1,2,\ldots,N$$

Successive Overrelaxation Method (SOR Method) / Überrelaxationsverfahren (SOR-Methode)

$$\{\mathbf{x}\}^{(l+1)} = \{\mathbf{x}\}^{(l)} - \omega\left[\overline{\{\mathbf{x}\}}^{(l+1)} - \{\mathbf{x}\}^{(l)}\right] \qquad l = 1, 2, \ldots, L$$

$$= (1-\omega)\{\mathbf{x}\}^{(l)} + \omega\overline{\{\mathbf{x}\}}^{(l+1)}$$

$$\{\mathbf{x}\}^{(l+1)} = \underbrace{\left([\mathbf{D}] + \omega[\mathbf{L}]\right)^{-1}\left[(1-\omega)[\mathbf{D}] - \omega[\mathbf{U}]\right]}_{=[\mathbf{G}]_{\mathrm{SOR}}}\{\mathbf{x}\}^{(l)} + \underbrace{\omega\left([\mathbf{D}] + \omega[\mathbf{L}]\right)^{-1}\{\mathbf{b}\}}_{=\{\mathbf{c}\}_{\mathrm{SOR}}} \qquad i = 1, 2, \ldots, N$$

$$[\mathbf{G}]_{\mathrm{SOR}} = \left([\mathbf{D}] + \omega[\mathbf{L}]\right)^{-1}\left[(1-\omega)[\mathbf{D}] - \omega[\mathbf{U}]\right]$$

$$\{\mathbf{c}\}_{\mathrm{SOR}} = \omega\left([\mathbf{D}] + \omega[\mathbf{L}]\right)^{-1}\{\mathbf{b}\}$$

$$\{\mathbf{x}\}^{(l+1)} = [\mathbf{G}]_{\mathrm{SOR}}\{\mathbf{x}\}^{(l)} + \{\mathbf{c}\}_{\mathrm{SOR}} \qquad l = 1, 2, \ldots, L$$

Symmetric Successive Overrelaxation Method (SSOR Method) / Symmetrische Überrelaxationsverfahren (SSOR-Methode)

Forward SOR step / Vorwärts-SOR-Schritt

$$x_i^{\left(l+\frac{1}{2}\right)} = (1-\omega)\, x_i^{(l)} + \omega \left\{ \sum_{j=1}^{i-1} G_{\mathrm{J},ij}\; x_j^{\left(l+\frac{1}{2}\right)} + \sum_{j=i+1}^{N} G_{\mathrm{J},ij}\, x_j^{(l)} + c_{\mathrm{J},i} \right\} \qquad i = 1, 2, \ldots, N$$

Backward SOR step / Rückwärts-SOR-Schritt

$$x_i^{(l+1)} = (1-\omega)\, x_i^{\left(l+\frac{1}{2}\right)} + \omega \left\{ \sum_{j=1}^{i-1} G_{\mathrm{J},ij}\; x_j^{\left(l+\frac{1}{2}\right)} + \sum_{j=i+1}^{N} G_{\mathrm{J},ij}\, x_j^{(l+1)} + c_{\mathrm{J},i} \right\} \qquad i = N, N-1, \ldots, 1$$

Forward SOR step / Vorwärts-SOR-Schritt

$$\{\mathbf{x}\}^{(l+\frac{1}{2})} = (1-\omega)\{\mathbf{x}\}^{(l)} - \omega \left\{ [\mathbf{D}]^{-1}[\mathbf{L}]\{\mathbf{x}\}^{(l+\frac{1}{2})} + [\mathbf{D}]^{-1}[\mathbf{U}]\{\mathbf{x}\}^{(l)} - [\mathbf{D}]^{-1}\{\mathbf{b}\} \right\} \qquad i = 1, 2, \ldots, N$$

Backward SOR step / Rückwärts-SOR-Schritt

$$\{\mathbf{x}\}^{(l+1)} = (1-\omega)\{\mathbf{x}\}^{(l+\frac{1}{2})} - \omega \left\{ [\mathbf{D}]^{-1}[\mathbf{L}]\{\mathbf{x}\}^{(l+\frac{1}{2})} + [\mathbf{D}]^{-1}[\mathbf{U}]\{\mathbf{x}\}^{(l)} - [\mathbf{D}]^{-1}\{\mathbf{b}\} \right\} \qquad i = N, N-1, \ldots, 1$$

Convergence of Point Iterative Methods / Konvergenz von punktiterativen Methoden

$$\rho\left([\mathbf{G}]\right) = \max_{n=1,2,\ldots,N} \left| \nu_n\left([\mathbf{G}]\right) \right|$$

$$\rho\left([\mathbf{G}]\right) < 1$$

$$0 < \omega_{\mathrm{SOR}} < 2$$

$$0 < \omega_{\mathrm{SSOR}} < 2$$

Error Vector and Error Measure / Fehlervektor und Fehlermaß

$$\omega_{\mathrm{SOR,opt}} = \frac{2}{1+\sqrt{1-\rho\left([\mathbf{G}]_{\mathrm{J}}\right)}}$$

$$= 1 + \left(\frac{\rho\left([\mathbf{G}]_{\mathrm{J}}\right)}{1+\sqrt{1-\rho\left([\mathbf{G}]_{\mathrm{J}}\right)}} \right)^2$$

$$\rho\left([\mathbf{G}]_{\mathrm{SOR}}\right) = \left(\frac{\rho\left([\mathbf{G}]_{\mathrm{J}}\right)}{1+\sqrt{1-\rho\left([\mathbf{G}]_{\mathrm{J}}\right)}} \right)^2$$

Symmetric positive definite [A] matrix /
Symmetrische positiv-definite [A] Matrix

$$\rho\left([\mathbf{L}][\mathbf{U}]\right) \le \frac{1}{4}$$

$$\omega_{\mathrm{SSOR}} = \frac{2}{1+2\sqrt{1-\rho\left([\mathbf{G}]_{\mathrm{J}}\right)}}$$

Approximation for a *D*-dimensional Dirichlet problem /
Approximation für ein *D*-dimensionales Dirichlet-Problem

$$\rho\left([\mathbf{G}]_{\mathrm{J}}\right) = \frac{\sum_{d=1}^{D} \cos\left(\frac{\pi}{N_d}\right)}{D}$$

Optimal Relaxation / Optimaler Relaxationsparameter

$$\{\mathbf{x}\}^{(l+1)} = [\mathbf{G}]\{\mathbf{x}\}^{(l)} + \{\mathbf{c}\}$$

$$\{\mathbf{x}\} = [\mathbf{G}]\{\mathbf{x}\} + \{\mathbf{c}\}$$

$$\{\boldsymbol{\varepsilon}\}^{(l)} = \{\mathbf{x}\}^{(l)} + \{\mathbf{x}\}$$

$$\{\boldsymbol{\varepsilon}\}^{(l+1)} = [\mathbf{G}]\{\boldsymbol{\varepsilon}\}^{(l)}$$

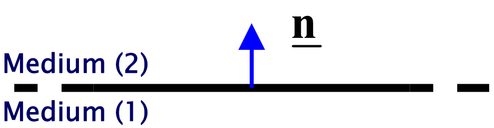
$$\lim_{l\to\infty}\{\boldsymbol{\varepsilon}\}^{(l)} = \{0\}$$

$$\varepsilon_{\text{max}}^{(\text{l}+1)} = \max_{n=1,2,\ldots,N}\left|\varepsilon^{(n,l+1)} - \varepsilon^{(n,l)}\right|$$

$$\varepsilon_{\text{rel,max}}^{(\text{l}+1)} = \max_{n=1,2,\ldots,N}\left|\frac{\varepsilon^{(n,l+1)} - \varepsilon^{(n,l)}}{\varepsilon^{(n,l+1)}}\right|$$

Electrostatic (ES) Fields / Elektrostatische (ES) Felder

Transition Conditions / Übergangsbedingungen



$$\underline{\mathbf{R}} \in S_{\mathrm{I(nterface)}}$$

$$E_{tan}^{(2)}\left(\underline{\mathbf{R}}\right) - E_{tan}^{(1)}\left(\underline{\mathbf{R}}\right) = 0$$

$$D_n^{(2)}\left(\underline{\mathbf{R}}\right) - D_n^{(2)}\left(\underline{\mathbf{R}}\right) = \eta_{\mathrm{e}}(\underline{\mathbf{R}})$$

$$E_{tan}^{(2)}\left(\underline{\mathbf{R}}\right) - E_{tan}^{(1)}\left(\underline{\mathbf{R}}\right) = 0$$

$$\Downarrow$$

$$\Phi_{\mathrm{e}}^{(2)}\left(\underline{\mathbf{R}}\right) - \Phi_{\mathrm{e}}^{(1)}\left(\underline{\mathbf{R}}\right) = \Phi_{\mathrm{e}0} = \mathrm{const.}$$

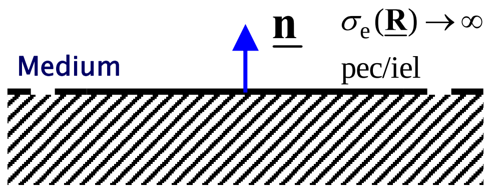
$$D_n^{(2)}\left(\underline{\mathbf{R}}\right) - D_n^{(2)}\left(\underline{\mathbf{R}}\right) = \eta_{\mathrm{e}}(\underline{\mathbf{R}})$$

$$\Downarrow$$

$$\frac{\partial}{\partial n}\Phi_{\mathrm{e}}^{(2)}\left(\underline{\mathbf{R}}\right) - \frac{\varepsilon_{\mathrm{r}}^{(1)}}{\varepsilon_{\mathrm{r}}^{(2)}}\frac{\partial}{\partial n}\Phi_{\mathrm{e}}^{(1)}\left(\underline{\mathbf{R}}\right) = -\frac{1}{\varepsilon_0\varepsilon_{\mathrm{r}}^{(2)}}\eta_{\mathrm{e}}(\underline{\mathbf{R}})$$

Boundary Conditions / Randbedingungen

$$\underline{\mathbf{R}} \in S_{\mathrm{B(oundary)}}$$



$$E_{tan}\left(\underline{\mathbf{R}}\right) = 0 \qquad \text{pec / iel}$$

$$D_n\left(\underline{\mathbf{R}}\right) = \eta_{\mathrm{e}}(\underline{\mathbf{R}}) \qquad \text{pec / iel}$$

$$E_{tan}\left(\underline{\mathbf{R}}\right) = 0$$

$$\Downarrow$$

$$\Phi_{\mathrm{e}}\left(\underline{\mathbf{R}}\right) = \Phi_{\mathrm{e}0} = \mathrm{const.}$$

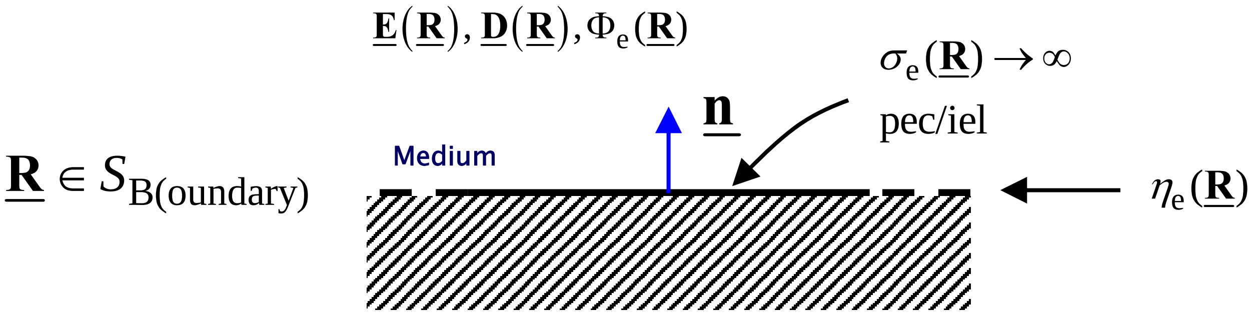
$$D_n\left(\underline{\mathbf{R}}\right) = \eta_{\mathrm{e}}(\underline{\mathbf{R}})$$

$$\Downarrow$$

$$\frac{\partial}{\partial n}\Phi_{\mathrm{e}}\left(\underline{\mathbf{R}}\right) = -\frac{1}{\varepsilon_0\varepsilon_{\mathrm{r}}}\eta_{\mathrm{e}}(\underline{\mathbf{R}})$$

Electrostatic (ES) Fields / Elektrostatische (ES) Felder

Boundary conditions for the electrostatic potential / Randbedingungen für die elektrostatische Potential



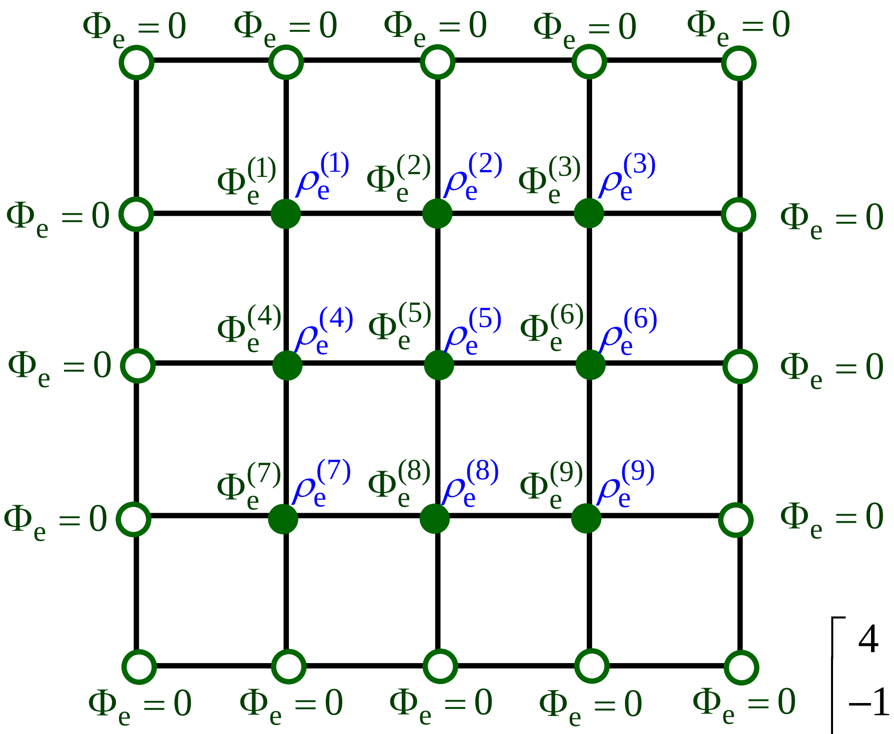
$$\Phi_e\left(\underline{\mathbf{R}}\right) = \Phi_{e0} = \text{const.} \quad (\Phi_{e0} = 0\ \mathrm{V})$$

$$\frac{\partial}{\partial n}\Phi_e\left(\underline{\mathbf{R}}\right) = -\frac{1}{\varepsilon_0 \varepsilon_r}\eta_e(\underline{\mathbf{R}})$$

Neumann Boundary Condition for Φ_e /
Neumann-Randbedingung für Φ_e

Dirichlet Boundary Condition for Φ_e
/
Dirichlet-Randbedingung für Φ_e

Example: 2-D Dirichlet Problem / Beispiel: 2D-Dirichlet-Problem



$$\underline{\mathbf{R}} = x\underline{\mathbf{e}}_x + z\underline{\mathbf{e}}_z$$

$$-\Delta\Phi_\mathrm{e}(\underline{\mathbf{R}}) = \frac{\rho_\mathrm{e}(\underline{\mathbf{R}})}{\varepsilon_0\varepsilon_\mathrm{r}} \qquad \underline{\mathbf{R}} \in S$$

$$-\oint\!\!\oint_{S=\partial V}\left[\nabla\Phi_\mathrm{e}(\underline{\mathbf{R}})\right]\bullet \underline{\mathbf{dS}} = \frac{1}{\varepsilon_0\varepsilon_\mathrm{r}}\iiint_V \rho_\mathrm{e}(\underline{\mathbf{R}})\,\mathrm{d}V$$

$$\underline{\mathbf{R}} \in S$$

Dirichlet boundary condition for Φ_e /
Dirichlet-Randbedingung für Φ_e

$$\Phi_\mathrm{e}(\underline{\mathbf{R}}) = 0 \qquad \underline{\mathbf{R}} \in C_D = \partial S$$

$$\begin{bmatrix} 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 \end{bmatrix} \begin{Bmatrix} \Phi_\mathrm{e}^{(1)} \\ \Phi_\mathrm{e}^{(2)} \\ \Phi_\mathrm{e}^{(3)} \\ \Phi_\mathrm{e}^{(4)} \\ \Phi_\mathrm{e}^{(5)} \\ \Phi_\mathrm{e}^{(6)} \\ \Phi_\mathrm{e}^{(7)} \\ \Phi_\mathrm{e}^{(8)} \\ \Phi_\mathrm{e}^{(9)} \end{Bmatrix} = \begin{Bmatrix} b^{(1)} \\ b^{(2)} \\ b^{(3)} \\ b^{(4)} \\ b^{(5)} \\ b^{(6)} \\ b^{(7)} \\ b^{(8)} \\ b^{(9)} \end{Bmatrix}$$

$$\underbrace{-\widetilde{[\mathbf{div}]}[\mathbf{grad}]}_{=[\mathbf{A}]}\underbrace{\{\Phi_\mathrm{e}\}}_{=\{\mathbf{x}\}} = \underbrace{\frac{(\Delta x)^2}{\varepsilon_0\varepsilon_\mathrm{r}}\{\boldsymbol{\rho}_\mathrm{e}\}}_{=\{\mathbf{b}\}}$$

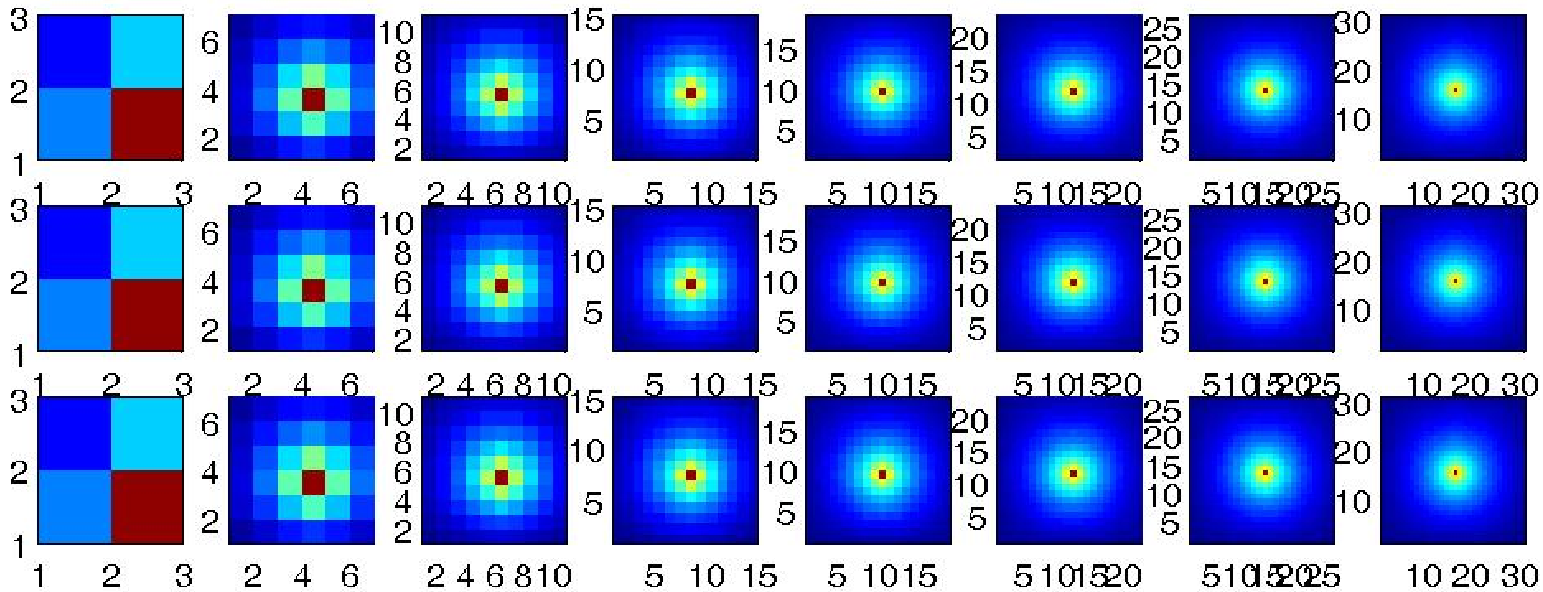
Dirichlet boundary condition for Φ_e /
Dirichlet-Randbedingung für Φ_e

$$x^{(n)} = \Phi_\mathrm{e}^{(n)} = 0 \qquad n \in C_D = \partial S$$

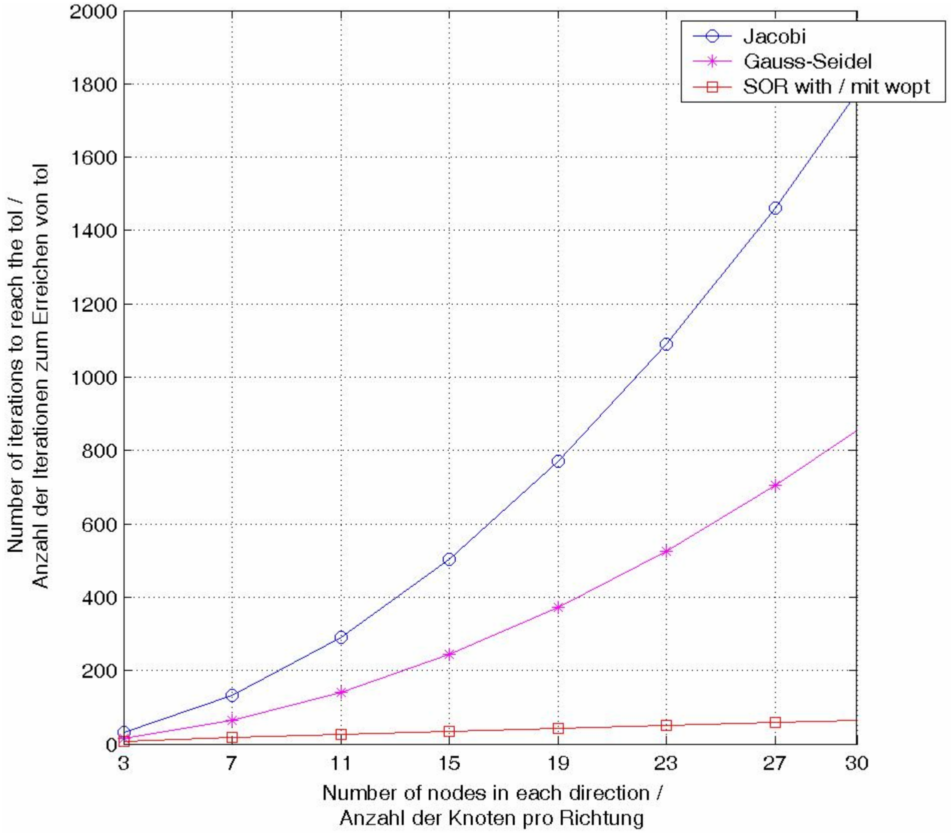
Example: 2-D Dirichlet Problem / Beispiel: 2D-Dirichlet-Problem

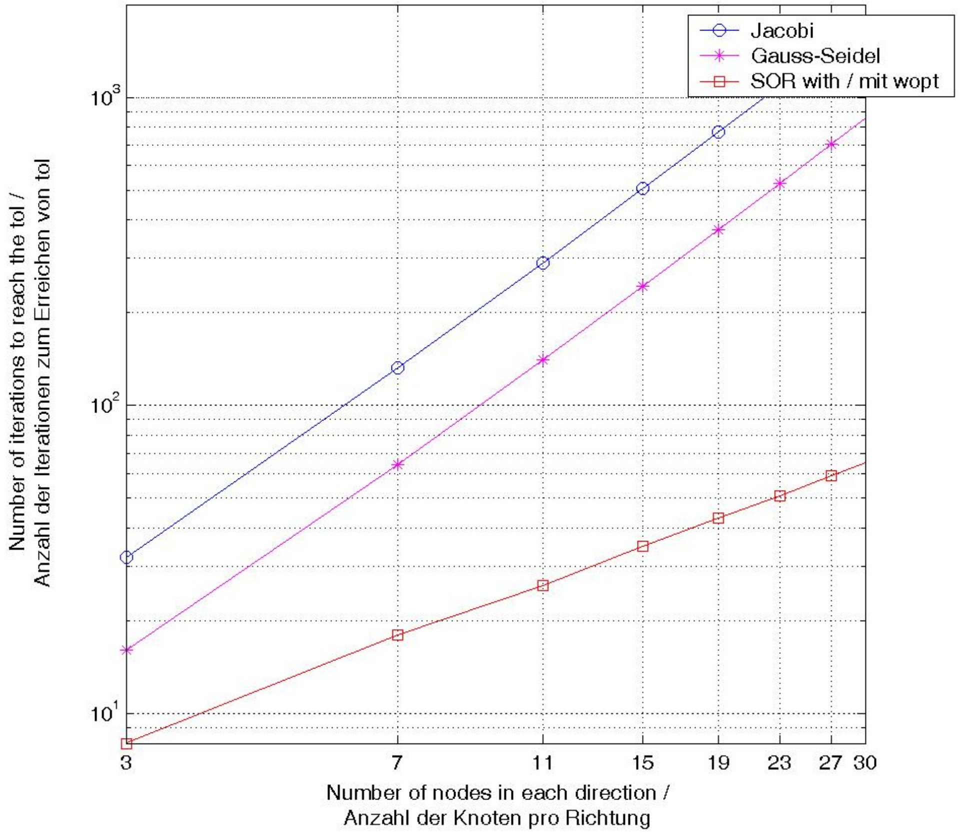
$$\rho_\mathrm{e}(x,z) = \xi_{\mathrm{e}0}\,\delta(x-x_\mathrm{s})\delta(z-z_\mathrm{s}) \qquad \underline{\mathbf{R}} = \underline{\mathbf{R}}_\mathrm{s} = x_\mathrm{s}\underline{\mathbf{e}}_x + z_\mathrm{s}\underline{\mathbf{e}}_z$$

$N=3$ $N=7$ $N=11$ $N=15$ $N=19$ $N=23$ $N=27$ $N=31$



Example: 2-D Dirichlet Problem / Beispiel: 2D-Dirichlet-Problem





End of Lecture 12 /
Ende der 12. Vorlesung