Numerical Methods of Electromagnetic Field Theory I (NFT I)
Numerische Methoden der Elektromagnetischen Feldtheorie I (NFT I) /

12th Lecture / 12. Vorlesung

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FIT Discretization of the 3rd and 4th Maxwell's Equation / FIT-Diskretisierung der 3. und 4. Maxwellschen Gleichung

Governing Analytic Equations

Maxwell's equations in integral form / Maxwellsche Gleichungen in Integralform

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint_S \underline{\mathbf{B}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = -\oint_{C=\partial S}\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dR}} - \iint_S \underline{\mathbf{J}}_\mathrm{m}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint_S \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = \oint_{C=\partial S}\underline{\mathbf{H}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dR}} - \iint_S \underline{\mathbf{J}}_\mathrm{e}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}}$$

$$\oiint_{S=\partial V}\underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = \iiint_V \rho_\mathrm{e}(\underline{\mathbf{R}},t)\,dV$$

$$\oiint_{S=\partial V}\underline{\mathbf{B}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = \iiint_V \rho_\mathrm{m}(\underline{\mathbf{R}},t)\,dV$$

FIT Grid Equations

Maxwell's grid equations / Maxwellsche Gittergleichungen

$$[\mathbf{S}]\frac{\mathrm{d}}{\mathrm{d}t}\{\mathbf{B}\}(t) = -[\mathbf{curl}][\mathbf{R}]\{\mathbf{E}\}(t) \quad -[\mathbf{S}]\{\mathbf{J}_\mathrm{m}\}(t)$$

$$[\widetilde{\boldsymbol{\varepsilon}}][\widetilde{\mathbf{S}}]\frac{\mathrm{d}}{\mathrm{d}t}\{\mathbf{E}\}(t) = [\widetilde{\mathbf{curl}}][\widetilde{\boldsymbol{\nu}}][\widetilde{\mathbf{R}}]\{\mathbf{B}\}(t) - [\widetilde{\mathbf{S}}]\{\mathbf{J}_\mathrm{e}\}(t)$$

$$[\widetilde{\mathbf{div}}][\widetilde{\boldsymbol{\varepsilon}}][\widetilde{\mathbf{S}}]\{\mathbf{E}\}(t) = [\widetilde{\mathbf{V}}]\{\boldsymbol{\rho}_\mathrm{e}\}(t) = \{\mathbf{Q}_\mathrm{e}\}(t)$$

$$[\mathbf{div}][\mathbf{S}]\{\mathbf{B}\}(t) = [\mathbf{V}]\{\boldsymbol{\rho}_\mathrm{m}\}(t) = \{\mathbf{Q}_\mathrm{m}\}(t)$$

3-D FIT – Electrostatic Case / 3D-FIT – Elektrostatischer Fall

Electric Gauss' grid equation – 3rd Maxwell's grid equation in global matrix form /
Elektrische Gaußsche Gittergleichung – 3. Maxwellsche Gittergleichung in globaler Matrixform

$$\widetilde{[\mathbf{div}]}\,\widetilde{[\boldsymbol{\varepsilon}]}\,\widetilde{[\mathbf{S}]}\{\mathbf{E}\}(t)=\widetilde{[\mathbf{V}]}\left\{\boldsymbol{\rho}_{\mathrm{e}}\right\}(t)=\left\{\mathbf{Q}_{\mathrm{e}}\right\}(t)\quad\xrightarrow{\frac{\partial}{\partial t}\equiv 0}\quad\widetilde{[\mathbf{div}]}\,\widetilde{[\boldsymbol{\varepsilon}]}\,\widetilde{[\mathbf{S}]}\{\mathbf{E}\}=\widetilde{[\mathbf{V}]}\left\{\boldsymbol{\rho}_{\mathrm{e}}\right\}=\left\{\mathbf{Q}_{\mathrm{e}}\right\}$$

$$\underline{\mathbf{E}}(\underline{\mathbf{R}})=-\nabla\Phi_{\mathrm{e}}(\underline{\mathbf{R}})$$

$$\begin{aligned}\underline{\mathbf{D}}(\underline{\mathbf{R}})&=\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\cdot\underline{\mathbf{E}}(\underline{\mathbf{R}})\\ \nabla\cdot\underline{\mathbf{D}}(\underline{\mathbf{R}})&=\rho_{\mathrm{e}}(\underline{\mathbf{R}})\\ &=\nabla\cdot\left[\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\cdot\underline{\mathbf{E}}(\underline{\mathbf{R}})\right]\\ &=\nabla\cdot\left\{\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\cdot\left[-\nabla\Phi_{\mathrm{e}}(\underline{\mathbf{R}})\right]\right\}\\ &=-\nabla\cdot\left\{\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\cdot\left[\nabla\Phi_{\mathrm{e}}(\underline{\mathbf{R}})\right]\right\}\end{aligned}$$

Inhomogeneous, anisotropic case /
Inhomogener anisotroper Fall

$$\nabla\cdot\left\{\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\cdot\left[\nabla\Phi_{\mathrm{e}}(\underline{\mathbf{R}})\right]\right\}=-\rho_{\mathrm{e}}(\underline{\mathbf{R}})$$

Homogeneous, isotropic case /
Homogener isotroper Fall

$$\underbrace{\nabla\cdot\nabla}_{=\Delta}\Phi_{\mathrm{e}}(\underline{\mathbf{R}})=-\frac{\rho_{\mathrm{e}}(\underline{\mathbf{R}})}{\varepsilon}$$

$$\Delta\Phi_{\mathrm{e}}(\underline{\mathbf{R}})=-\frac{\rho_{\mathrm{e}}(\underline{\mathbf{R}})}{\varepsilon}$$

3-D FIT – Electrostatic Case / 3D-FIT – Elektrostatischer Fall

$$\underline{\mathbf{E}}(\underline{\mathbf{R}})=-\nabla\Phi_{\mathrm{e}}(\underline{\mathbf{R}})$$

$$\begin{aligned}\underline{\mathbf{D}}(\underline{\mathbf{R}})&=\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\cdot\underline{\mathbf{E}}(\underline{\mathbf{R}})\\ \nabla\cdot\underline{\mathbf{D}}(\underline{\mathbf{R}})&=\rho_{\mathrm{e}}(\underline{\mathbf{R}})\\ &=\nabla\cdot\left[\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\cdot\underline{\mathbf{E}}(\underline{\mathbf{R}})\right]\\ &=\nabla\cdot\left\{\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\cdot\left[-\nabla\Phi_{\mathrm{e}}(\underline{\mathbf{R}})\right]\right\}\\ &=-\nabla\cdot\left\{\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\cdot\left[\nabla\Phi_{\mathrm{e}}(\underline{\mathbf{R}})\right]\right\}\end{aligned}$$

$$\begin{aligned}\underline{\mathbf{D}}(\underline{\mathbf{R}})&=\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\cdot\underline{\mathbf{E}}(\underline{\mathbf{R}})\\ \oiint_{S=\partial V}\underline{\mathbf{D}}(\underline{\mathbf{R}})\cdot\underline{\mathbf{dS}}&=\iiint_V\rho_{\mathrm{e}}(\underline{\mathbf{R}})\,\mathrm{d}V\\ &=\oiint_{S=\partial V}\left[\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\cdot\underline{\mathbf{E}}(\underline{\mathbf{R}})\right]\cdot\underline{\mathbf{dS}}\\ &=\oiint_{S=\partial V}\left\{\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\cdot\left[-\nabla\Phi_{\mathrm{e}}(\underline{\mathbf{R}})\right]\right\}\cdot\underline{\mathbf{dS}}\nabla\cdot\\ &=-\oiint_{S=\partial V}\left\{\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\cdot\left[\nabla\Phi_{\mathrm{e}}(\underline{\mathbf{R}})\right]\right\}\cdot\underline{\mathbf{dS}}\nabla\cdot\end{aligned}$$

$$-\oiint_{S=\partial V}\left\{\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\cdot\left[\nabla\Phi_{\mathrm{e}}(\underline{\mathbf{R}})\right]\right\}\cdot\underline{\mathbf{dS}}=\iiint_V\rho_{\mathrm{e}}(\underline{\mathbf{R}})\,\mathrm{d}V$$

Inhomogeneous, anisotropic case /
Inhomogener anisotroper Fall

$$\nabla\cdot\left\{\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\cdot\left[\nabla\Phi_{\mathrm{e}}(\underline{\mathbf{R}})\right]\right\}=-\rho_{\mathrm{e}}(\underline{\mathbf{R}})$$

Homogeneous, isotropic case /
Homogener isotroper Fall

$$\underbrace{\nabla\cdot\nabla}_{=\Delta}\Phi_{\mathrm{e}}(\underline{\mathbf{R}})=-\frac{\rho_{\mathrm{e}}(\underline{\mathbf{R}})}{\varepsilon}$$

$$\Delta\Phi_{\mathrm{e}}(\underline{\mathbf{R}})=-\frac{\rho_{\mathrm{e}}(\underline{\mathbf{R}})}{\varepsilon}$$

FIT Discretization of Scalar Electric Potential / FIT-Diskretisierung des skalaren elektrischen Potentials

Differential form / Differentialform

$$\underline{\mathbf{E}}(\underline{\mathbf{R}}) = -\nabla\Phi_\mathrm{e}(\underline{\mathbf{R}})$$

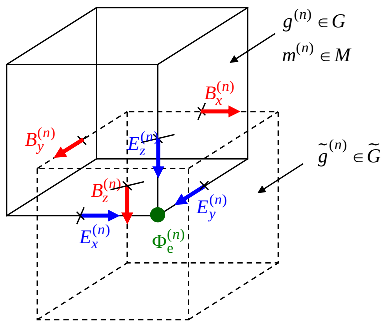
Integral form / Integralform

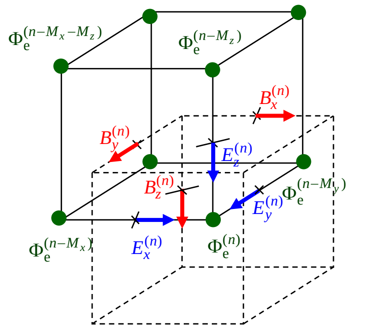
$$\int_C \underline{\mathbf{E}}(\underline{\mathbf{R}})\cdot \mathbf{d}\underline{\mathbf{R}} = -\int_C \nabla\Phi_\mathrm{e}(\underline{\mathbf{R}})\cdot \mathbf{d}\underline{\mathbf{R}}$$
$$= -\left[\Phi_\mathrm{e}(\underline{\mathbf{R}}_2) - \Phi_\mathrm{e}(\underline{\mathbf{R}}_1)\right]$$

FIT grid equation / FIT-Gittergleichung

$$\{E\}^{(n)} = -[R]^{-1}[\mathrm{grad}]\{\Phi_\mathrm{e}\}^{(n)}$$

$$\{\mathbf{E}\} = -[\mathbf{R}]^{-1}[\mathbf{grad}]\{\Phi_\mathrm{e}\}$$





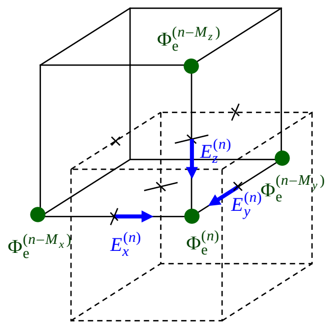
FIT Discretization of Scalar Electric Potential / FIT-Diskretisierung des skalaren elektrischen Potentials

Integral form / Integralform

$$\int_C \underline{\mathbf{E}}(\underline{\mathbf{R}})\cdot \mathbf{d}\underline{\mathbf{R}} = -\int_C \left[\nabla\,\Phi_\mathrm{e}(\underline{\mathbf{R}})\right]\cdot \mathbf{d}\underline{\mathbf{R}}$$
$$= -\left[\Phi_\mathrm{e}(\underline{\mathbf{R}}_2) - \Phi_\mathrm{e}(\underline{\mathbf{R}}_1)\right]$$

$$\int_C \underline{\mathbf{E}}(\underline{\mathbf{R}})\cdot \mathbf{d}\underline{\mathbf{R}} = \int_{x=x_0}^{x_0+\Delta x} \underline{\mathbf{E}}(x,y,z)\cdot \mathbf{d}\underline{\mathbf{R}}$$
$$= \int_{x=x_0}^{x_0+\Delta x} \underline{\mathbf{E}}(x,y,z)\cdot \underline{\mathrm{e}}_x\,\mathrm{d}x$$
$$= \int_{x=x_0}^{x_0+\Delta x} E_x(x,y,z)\,\mathrm{d}x$$
$$= E_x^{(n)} \int_{x=x_0}^{x_0+\Delta x} \mathrm{d}x$$
$$= E_x^{(n)}\Delta x$$

$$\int_C \left[\nabla\,\Phi_\mathrm{e}(\underline{\mathbf{R}})\right]\cdot \mathbf{d}\underline{\mathbf{R}} = \int_{x=x_0}^{x_0+\Delta x} \left[\nabla\,\Phi_\mathrm{e}(x,y,z)\right]\cdot \mathbf{d}\underline{\mathbf{R}}$$
$$= \int_{x=x_0}^{x_0+\Delta x} \left[\nabla\,\Phi_\mathrm{e}(x,y,z)\right]\cdot \underline{\mathrm{e}}_x\,\mathrm{d}x$$
$$= \int_{x=x_0}^{x_0+\Delta x} \frac{\partial}{\partial x}\Phi_\mathrm{e}(x,y,z)\,\mathrm{d}x$$
$$= \Phi_\mathrm{e}(x_0,y,z) - \Phi_\mathrm{e}(x_0+\Delta x,y,z)$$
$$= \Phi_\mathrm{e}^{(n-M_x)} - \Phi_\mathrm{e}^{(n)}$$



$$\int_C \underline{\mathbf{E}}(\underline{\mathbf{R}})\cdot \mathbf{d}\underline{\mathbf{R}} = -\int_C \left[\nabla\,\Phi_\mathrm{e}(\underline{\mathbf{R}})\right]\cdot \mathbf{d}\underline{\mathbf{R}}$$
$$= -\left[\Phi_\mathrm{e}(\underline{\mathbf{R}}_2) - \Phi_\mathrm{e}(\underline{\mathbf{R}}_1)\right]$$

$$E_x^{(n)}\Delta x = -\Phi_\mathrm{e}^{(n)} - \Phi_\mathrm{e}^{(n-M_x)} = -(I - S_{-M_x})\,\Phi_\mathrm{e}^{(n)}$$
$$E_y^{(n)}\Delta y = -\Phi_\mathrm{e}^{(n)} - \Phi_\mathrm{e}^{(n-M_y)} = -(I - S_{-M_y})\,\Phi_\mathrm{e}^{(n)}$$
$$E_z^{(n)}\Delta z = -\Phi_\mathrm{e}^{(n)} - \Phi_\mathrm{e}^{(n-M_z)} = -(I - S_{-M_z})\,\Phi_\mathrm{e}^{(n)}$$

FIT Discretization of Scalar Electric Potential / FIT-Diskretisierung des skalaren elektrischen Potentials

$$\int_C \underline{\mathbf{E}}(\underline{\mathbf{R}}) \cdot \underline{\mathbf{dR}} = -\int_C \left[\nabla \Phi_e(\underline{\mathbf{R}})\right] \cdot \underline{\mathbf{dR}} = -\left[\Phi_e(\underline{\mathbf{R}}_2) - \Phi_e(\underline{\mathbf{R}}_1)\right]$$

$$\underbrace{\begin{bmatrix} \Delta x & & \\ & \Delta y & \\ & & \Delta z \end{bmatrix}}_{=[R]} \underbrace{\begin{Bmatrix} E_x \\ E_y \\ E_z \end{Bmatrix}^{(n)}}_{=\{E\}^{(n)}} = \underbrace{-\begin{bmatrix} (I - S_{-M_x}) \\ (I - S_{-M_y}) \\ (I - S_{-M_z}) \end{bmatrix}}_{=-[\mathrm{grad}]} \Phi_e^{(n)}$$

$$E_x^{(n)} \Delta x = -\Phi_e^{(n)} - \Phi_e^{(n-M_x)} = -(I - S_{-M_x})\, \Phi_e^{(n)}$$

$$E_y^{(n)} \Delta y = -\Phi_e^{(n)} - \Phi_e^{(n-M_y)} = -(I - S_{-M_y})\, \Phi_e^{(n)}$$

$$E_z^{(n)} \Delta z = -\Phi_e^{(n)} - \Phi_e^{(n-M_z)} = -(I - S_{-M_z})\, \Phi_e^{(n)}$$

$$[R]\{E\}^{(n)} = -[\mathrm{grad}]\Phi_e^{(n)}$$

$$[R] = \begin{bmatrix} \Delta x & & \\ & \Delta y & \\ & & \Delta z \end{bmatrix} \to [R]^{-1} = \begin{bmatrix} \frac{1}{\Delta x} & & \\ & \frac{1}{\Delta y} & \\ & & \frac{1}{\Delta z} \end{bmatrix}$$

$$\{E\}^{(n)} = -[R]^{-1}[\mathrm{grad}]\Phi_e^{(n)}$$

3-D FIT - Electrostatic Case / 3D-FIT - Elektrostatischer Fall

Electrostatic Poisson's grid equation / Elektrostatische Poissonsche Gittergleichung

$$\underline{\mathbf{E}}(\underline{\mathbf{R}}) = -\nabla \Phi_e(\underline{\mathbf{R}})$$

$$\{\mathbf{E}\} = -[\mathbf{R}]^{-1}[\mathbf{grad}]\{\Phi_e\}$$

Inhomogeneous, anisotropic case / Inhomogener anisotroper Fall

$$[\widetilde{\mathbf{div}}][\boldsymbol{\varepsilon}][\widetilde{\mathbf{S}}]\{\mathbf{E}\} = -[\widetilde{\mathbf{div}}][\boldsymbol{\varepsilon}][\widetilde{\mathbf{S}}][\mathbf{R}]^{-1}[\mathbf{grad}]\{\Phi_e\}$$

$$\nabla \cdot \left\{\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}}) \cdot \left[\nabla \Phi_e(\underline{\mathbf{R}})\right]\right\} = -\rho_e(\underline{\mathbf{R}})$$

$$[\widetilde{\mathbf{div}}][\boldsymbol{\varepsilon}][\widetilde{\mathbf{S}}][\mathbf{R}]^{-1}[\mathbf{grad}]\{\Phi_e\} = -[\widetilde{\mathbf{V}}]\{\boldsymbol{\rho}_e\}$$

Homogeneous, isotropic case / Homogener isotroper Fall

$$\Downarrow$$

$$\underbrace{\nabla \cdot \nabla}_{=\Delta} \Phi_e(\underline{\mathbf{R}}) = -\frac{\rho_e(\underline{\mathbf{R}})}{\varepsilon}$$

$$[\mathbf{A}]\{\mathbf{x}\} = \{\mathbf{b}\}$$

with / mit

$$\Delta \Phi_e(\underline{\mathbf{R}}) = -\frac{\rho_e(\underline{\mathbf{R}})}{\varepsilon}$$

$$[\mathbf{A}] = [\widetilde{\mathbf{div}}][\boldsymbol{\varepsilon}][\widetilde{\mathbf{S}}][\mathbf{R}]^{-1}[\mathbf{grad}]$$

$$\{\mathbf{x}\} = \{\Phi_e\}$$

$$\{\mathbf{b}\} = -[\widetilde{\mathbf{V}}]\{\boldsymbol{\rho}_e\}$$

3-D FIT – Electrostatic Case / 3D-FIT – Elektrostatischer Fall

Electrostatic Poisson's grid equation / Elektrostatische Poissonsche Gittergleichung

$$[\widetilde{\mathbf{div}}][\widetilde{\boldsymbol{\varepsilon}}][\widetilde{\mathbf{S}}][\mathbf{R}]^{-1}[\mathbf{grad}]\{\Phi_\mathrm{e}\} = -[\widetilde{\mathbf{V}}]\{\boldsymbol{\rho}_\mathrm{e}\}$$

$$[\widetilde{\mathbf{S}}] = \begin{bmatrix} [\mathrm{diag}\{\Delta y \Delta z\}]_{N\times N} & [0] & [0] \\ [0] & [\mathrm{diag}\{\Delta x \Delta z\}]_{N\times N} & [0] \\ [0] & [0] & [\mathrm{diag}\{\Delta x \Delta y\}]_{N\times N} \end{bmatrix}_{3N\times 3N} \qquad \{\Phi_\mathrm{e}\} = \begin{Bmatrix} \Phi_\mathrm{e}^{(1)}(t) \\ \Phi_\mathrm{e}^{(2)}(t) \\ \vdots \\ \Phi_\mathrm{e}^{(N)}(t) \end{Bmatrix}_N \quad i = x, y, z$$

$$[\mathbf{R}] = \begin{bmatrix} [\mathrm{diag}\{\Delta x\}]_{N\times N} & [0] & [0] \\ [0] & [\mathrm{diag}\{\Delta y\}]_{N\times N} & [0] \\ [0] & [0] & [\mathrm{diag}\{\Delta z\}]_{N\times N} \end{bmatrix}_{3N\times 3N}$$

$$[\mathbf{R}]^{-1} = \begin{bmatrix} \left[\mathrm{diag}\left\{\frac{1}{\Delta x}\right\}\right]_{N\times N} & [0] & [0] \\ [0] & \left[\mathrm{diag}\left\{\frac{1}{\Delta y}\right\}\right]_{N\times N} & [0] \\ [0] & [0] & \left[\mathrm{diag}\left\{\frac{1}{\Delta z}\right\}\right]_{N\times N} \end{bmatrix}_{3N\times 3N}$$

3-D FIT – Electrostatic Case / 3D-FIT – Elektrostatischer Fall

Electrostatic Poisson's grid equation / Elektrostatische Poissonsche Gittergleichung

$$[\widetilde{\mathbf{div}}][\widetilde{\boldsymbol{\varepsilon}}][\widetilde{\mathbf{S}}][\mathbf{R}]^{-1}[\mathbf{grad}]\{\Phi_\mathrm{e}\} = -[\widetilde{\mathbf{V}}]\{\boldsymbol{\rho}_\mathrm{e}\}$$

$$[\widetilde{\mathbf{S}}][\mathbf{R}]^{-1} = \begin{bmatrix} [\mathrm{diag}\{\Delta y \Delta z\}]_{N\times N} & [0] & [0] \\ [0] & [\mathrm{diag}\{\Delta x \Delta z\}]_{N\times N} & [0] \\ [0] & [0] & [\mathrm{diag}\{\Delta x \Delta y\}]_{N\times N} \end{bmatrix} \begin{bmatrix} \left[\mathrm{diag}\left\{\frac{1}{\Delta x}\right\}\right]_{N\times N} & [0] & [0] \\ [0] & \left[\mathrm{diag}\left\{\frac{1}{\Delta y}\right\}\right]_{N\times N} & [0] \\ [0] & [0] & \left[\mathrm{diag}\left\{\frac{1}{\Delta z}\right\}\right]_{N\times N} \end{bmatrix}$$

$$= \begin{bmatrix} \left[\mathrm{diag}\left\{\frac{\Delta y \Delta z}{\Delta x}\right\}\right]_{N\times N} & [0] & [0] \\ [0] & \left[\mathrm{diag}\left\{\frac{\Delta x \Delta z}{\Delta y}\right\}\right]_{N\times N} & [0] \\ [0] & [0] & \left[\mathrm{diag}\left\{\frac{\Delta x \Delta y}{\Delta z}\right\}\right]_{N\times N} \end{bmatrix}$$

3-D FIT - Electrostatic Case / 3D-FIT - Elektrostatischer Fall

$$\oint\!\!\!\oint_{S=\partial V} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = \iiint_V \rho_{\mathrm{e}}(\underline{\mathbf{R}},t)\,dV$$

$$\oint\!\!\!\oint_{S=\partial V} \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = \left(S_{M_x}-I\right)D_x^{(n)}(t)\Delta y\Delta z + \left(S_{M_y}-I\right)D_y^{(n)}(t)\Delta x\Delta z + \left(S_{M_z}-I\right)D_z^{(n)}(t)\Delta x\Delta y$$

$$\oint\!\!\!\oint_{S=\partial V} \left[\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\right]\bullet\underline{\mathbf{dS}} = \left(S_{M_x}-I\right)\tilde{\varepsilon}_{xx}^{(n)}E_x^{(n)}(t)\Delta y\Delta z + \left(S_{M_y}-I\right)\tilde{\varepsilon}_{yy}^{(n)}E_y^{(n)}(t)\Delta x\Delta z + \left(S_{M_z}-I\right)\tilde{\varepsilon}_{zz}^{(n)}E_z^{(n)}(t)\Delta x\Delta y$$

$$\iiint_V \rho_{\mathrm{e}}(\underline{\mathbf{R}},t)\,dV = \rho_{\mathrm{e}}^{(n)}(t)\Delta x\Delta y\Delta z$$

$$E_x^{(n)} = -\frac{1}{\Delta x}\left(\Phi_{\mathrm{e}}^{(n)} - \Phi_{\mathrm{e}}^{(n-M_x)}\right) = -\frac{1}{\Delta x}(I - S_{-M_x})\,\Phi_{\mathrm{e}}^{(n)}$$

$$E_y^{(n)} = -\frac{1}{\Delta y}\left(\Phi_{\mathrm{e}}^{(n)} - \Phi_{\mathrm{e}}^{(n-M_y)}\right) = -\frac{1}{\Delta y}(I - S_{-M_y})\,\Phi_{\mathrm{e}}^{(n)}$$

$$E_z^{(n)} = -\frac{1}{\Delta z}\left(\Phi_{\mathrm{e}}^{(n)} - \Phi_{\mathrm{e}}^{(n-M_z)}\right) = -\frac{1}{\Delta z}(I - S_{-M_z})\,\Phi_{\mathrm{e}}^{(n)}$$

$$\begin{aligned}\oint\!\!\!\oint_{S=\partial V} \left[\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\right]\bullet\underline{\mathbf{dS}} &= \left(S_{M_x}-I\right)\tilde{\varepsilon}_{xx}^{(n)}E_x^{(n)}\Delta y\Delta z + \left(S_{M_y}-I\right)\tilde{\varepsilon}_{yy}^{(n)}E_y^{(n)}\Delta x\Delta z + \left(S_{M_z}-I\right)\tilde{\varepsilon}_{zz}^{(n)}E_z^{(n)}\Delta x\Delta y\\
&= \left(S_{M_x}-I\right)\tilde{\varepsilon}_{xx}^{(n)}\left[-\frac{1}{\Delta x}(I-S_{-M_x})\,\Phi_{\mathrm{e}}^{(n)}\right]\Delta y\Delta z\\
&\quad + \left(S_{M_y}-I\right)\tilde{\varepsilon}_{yy}^{(n)}\left[-\frac{1}{\Delta y}(I-S_{-M_y})\,\Phi_{\mathrm{e}}^{(n)}\right]\Delta x\Delta z\\
&\quad + \left(S_{M_z}-I\right)\tilde{\varepsilon}_{zz}^{(n)}\left[-\frac{1}{\Delta z}(I-S_{-M_z})\,\Phi_{\mathrm{e}}^{(n)}\right]\Delta x\Delta y\end{aligned}$$

3-D FIT - Electrostatic Case / 3D-FIT - Elektrostatischer Fall

$$\iiint_V \rho_{\mathrm{e}}(\underline{\mathbf{R}},t)\,dV = \rho_{\mathrm{e}}^{(n)}(t)\Delta x\Delta y\Delta z$$

$$\begin{aligned}\oint\!\!\!\oint_{S=\partial V} \left[\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\right]\bullet\underline{\mathbf{dS}} &= \left(S_{M_x}-I\right)\tilde{\varepsilon}_{xx}^{(n)}\left[-\frac{1}{\Delta x}(I-S_{-M_x})\,\Phi_{\mathrm{e}}^{(n)}\right]\Delta y\Delta z\\
&\quad + \left(S_{M_y}-I\right)\tilde{\varepsilon}_{yy}^{(n)}\left[-\frac{1}{\Delta y}(I-S_{-M_y})\,\Phi_{\mathrm{e}}^{(n)}\right]\Delta x\Delta z\\
&\quad + \left(S_{M_z}-I\right)\tilde{\varepsilon}_{zz}^{(n)}\left[-\frac{1}{\Delta z}(I-S_{-M_z})\,\Phi_{\mathrm{e}}^{(n)}\right]\Delta x\Delta y\end{aligned}$$

$$\begin{aligned}&\left(S_{M_x}-I\right)\tilde{\varepsilon}_{xx}^{(n)}\left[-\frac{1}{\Delta x}(I-S_{-M_x})\,\Phi_{\mathrm{e}}^{(n)}\right]\Delta y\Delta z\\
&+\left(S_{M_y}-I\right)\tilde{\varepsilon}_{yy}^{(n)}\left[-\frac{1}{\Delta y}(I-S_{-M_y})\,\Phi_{\mathrm{e}}^{(n)}\right]\Delta x\Delta z\\
&+\left(S_{M_z}-I\right)\tilde{\varepsilon}_{zz}^{(n)}\left[-\frac{1}{\Delta z}(I-S_{-M_z})\,\Phi_{\mathrm{e}}^{(n)}\right]\Delta x\Delta y = \rho_{\mathrm{e}}^{(n)}(t)\Delta x\Delta y\Delta z\end{aligned}$$

$$\begin{aligned}&-\frac{1}{(\Delta x)^2}\left(S_{M_x}-I\right)\tilde{\varepsilon}_{xx}^{(n)}(I-S_{-M_x})\,\Phi_{\mathrm{e}}^{(n)}\\
&-\frac{1}{(\Delta y)^2}\left(S_{M_y}-I\right)\tilde{\varepsilon}_{yy}^{(n)}(I-S_{-M_y})\,\Phi_{\mathrm{e}}^{(n)}\\
&-\frac{1}{(\Delta z)^2}\left(S_{M_z}-I\right)\tilde{\varepsilon}_{zz}^{(n)}(I-S_{-M_z})\,\Phi_{\mathrm{e}}^{(n)} = \rho_{\mathrm{e}}^{(n)}\end{aligned}$$

3-D FIT – Electrostatic Case / 3D-FIT – Elektrostatischer Fall

$$
\begin{aligned}
\frac{1}{(\Delta x)^2}\left(S_{M_x}-I\right)\tilde{\varepsilon}_{xx}^{(n)}(I-S_{-M_x})\,\Phi_{\mathrm{e}}^{(n)} &= \frac{1}{(\Delta x)^2}\left(S_{M_x}-I\right)(\tilde{\varepsilon}_{xx}^{(n)}-\tilde{\varepsilon}_{xx}^{(n)}S_{-M_x}\Phi_{\mathrm{e}}^{(n)}) \\
&= \frac{1}{(\Delta x)^2}\left[S_{M_x}\tilde{\varepsilon}_{xx}^{(n)}\Phi_{\mathrm{e}}^{(n)} - \underbrace{S_{M_x}\tilde{\varepsilon}_{xx}^{(n)}S_{-M_x}\Phi_{\mathrm{e}}^{(n)}}_{=\tilde{\varepsilon}_{xx}^{(n+M_x)}\underbrace{S_{M_x}S_{-M_x}}_{=I}\Phi_{\mathrm{e}}^{(n)}} - \tilde{\varepsilon}_{xx}^{(n)}\Phi_{\mathrm{e}}^{(n)} + \tilde{\varepsilon}_{xx}^{(n)}S_{-M_x}\Phi_{\mathrm{e}}^{(n)}\right] \\
&= \frac{1}{(\Delta x)^2}\left[\tilde{\varepsilon}_{xx}^{(n+M_x)}\Phi_{\mathrm{e}}^{(n+M_x)} - \underbrace{\tilde{\varepsilon}_{xx}^{(n+M_x)}\Phi_{\mathrm{e}}^{(n)} - \tilde{\varepsilon}_{xx}^{(n)}\Phi_{\mathrm{e}}^{(n)}}_{=\left[\tilde{\varepsilon}_{xx}^{(n)}+\tilde{\varepsilon}_{xx}^{(n+M_x)}\right]\Phi_{\mathrm{e}}^{(n)}} + \tilde{\varepsilon}_{xx}^{(n)}\Phi_{\mathrm{e}}^{(n-M_x)}\right] \\
&= \frac{1}{(\Delta x)^2}\left\{\tilde{\varepsilon}_{xx}^{(n+M_x)}S_{M_x} - \left[\underbrace{(I+S_{M_x})\tilde{\varepsilon}_{xx}^{(n)}}_{=2A_{M_x}}\right]I + \tilde{\varepsilon}_{xx}^{(n)}S_{-M_x}\right\}\Phi_{\mathrm{e}}^{(n)} \\
&= \frac{1}{(\Delta x)^2}\left\{\tilde{\varepsilon}_{xx}^{(n+M_x)}S_{M_x} - \left[2A_{M_x}\tilde{\varepsilon}_{xx}^{(n)}\right]I + \tilde{\varepsilon}_{xx}^{(n)}S_{-M_x}\right\}\Phi_{\mathrm{e}}^{(n)}
\end{aligned}
$$

$$
\frac{1}{(\Delta x)^2}\left(S_{M_x}-I\right)\tilde{\varepsilon}_{xx}^{(n)}(I-S_{-M_x})\,\Phi_{\mathrm{e}}^{(n)} = \frac{1}{(\Delta x)^2}\left\{\tilde{\varepsilon}_{xx}^{(n+M_x)}S_{M_x} - \left[2A_{M_x}\tilde{\varepsilon}_{xx}^{(n)}\right]I + \tilde{\varepsilon}_{xx}^{(n)}S_{-M_x}\right\}\Phi_{\mathrm{e}}^{(n)}
$$

3-D FIT – Electrostatic Case / 3D-FIT – Elektrostatischer Fall

$$
\begin{aligned}
\frac{1}{(\Delta x)^2}\left(S_{M_x}-I\right)\tilde{\varepsilon}_{xx}^{(n)}(I-S_{-M_x})\,\Phi_{\mathrm{e}}^{(n)} &= \frac{1}{(\Delta x)^2}\left\{\tilde{\varepsilon}_{xx}^{(n+M_x)}S_{M_x} - \left[2A_{M_x}\tilde{\varepsilon}_{xx}^{(n)}\right]I + \tilde{\varepsilon}_{xx}^{(n)}S_{-M_x}\right\}\Phi_{\mathrm{e}}^{(n)} \\
&= \left\{\frac{\tilde{\varepsilon}_{xx}^{(n+M_x)}}{(\Delta x)^2}S_{M_x} - \frac{\left[2A_{M_x}\tilde{\varepsilon}_{xx}^{(n)}\right]}{(\Delta x)^2}I + \frac{\tilde{\varepsilon}_{xx}^{(n)}}{(\Delta x)^2}S_{-M_x}\right\}\Phi_{\mathrm{e}}^{(n)}
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{(\Delta y)^2}\left(S_{M_y}-I\right)\tilde{\varepsilon}_{yy}^{(n)}(I-S_{-M_y})\,\Phi_{\mathrm{e}}^{(n)} &= \frac{1}{(\Delta y)^2}\left\{\tilde{\varepsilon}_{yy}^{(n+M_y)}\Phi_{\mathrm{e}}^{(n+M_y)} - \left[2S_{M_y}\tilde{\varepsilon}_{yy}^{(n)}\right]\Phi_{\mathrm{e}}^{(n)} + \tilde{\varepsilon}_{yy}^{(n)}\Phi_{\mathrm{e}}^{(n-M_y)}\right\} \\
&= \left\{\frac{\tilde{\varepsilon}_{yy}^{(n+M_y)}}{(\Delta y)^2}S_{M_y} - \frac{\left[2A_{M_y}\tilde{\varepsilon}_{yy}^{(n)}\right]}{(\Delta y)^2}I + \frac{\tilde{\varepsilon}_{yy}^{(n)}}{(\Delta y)^2}S_{-M_y}\right\}\Phi_{\mathrm{e}}^{(n+)}
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{(\Delta z)^2}\left(S_{M_z}-I\right)\tilde{\varepsilon}_{zz}^{(n)}(I-S_{-M_z})\,\Phi_{\mathrm{e}}^{(n)} &= \frac{1}{(\Delta z)^2}\left\{\tilde{\varepsilon}_{zz}^{(n+M_z)}\Phi_{\mathrm{e}}^{(n+M_z)} - \left[(I-S_{M_z})\tilde{\varepsilon}_{zz}^{(n)}\right]\Phi_{\mathrm{e}}^{(n)} + \tilde{\varepsilon}_{zz}^{(n)}\Phi_{\mathrm{e}}^{(n-M_z)}\right\} \\
&= \left\{\frac{\tilde{\varepsilon}_{zz}^{(n+M_z)}}{(\Delta z)^2}S_{M_z} - \frac{\left[2A_{M_z}\tilde{\varepsilon}_{zz}^{(n)}\right]}{(\Delta z)^2}I + \frac{\tilde{\varepsilon}_{zz}^{(n)}}{(\Delta z)^2}S_{-M_z}\right\}\Phi_{\mathrm{e}}^{(n)}
\end{aligned}
$$

3-D FIT – Electrostatic Case / 3D-FIT – Elektrostatischer Fall

$$\frac{1}{(\Delta x)^2}\left(S_{M_x}-I\right)\tilde{\varepsilon}_{xx}^{(n)}(I-S_{-M_x})\,\Phi_{\mathrm{e}}^{(n)}+\frac{1}{(\Delta z)^2}\left(S_{M_z}-I\right)\tilde{\varepsilon}_{zz}^{(n)}(I-S_{-M_z})\,\Phi_{\mathrm{e}}^{(n)}+\frac{1}{(\Delta y)^2}\left(S_{M_y}-I\right)\tilde{\varepsilon}_{yy}^{(n)}(I-S_{-M_y})\,\Phi_{\mathrm{e}}^{(n)}$$

$$=\left\{\frac{\tilde{\varepsilon}_{xx}^{(n+M_x)}}{(\Delta x)^2}S_{M_x}-\frac{\left[2A_{M_x}\tilde{\varepsilon}_{xx}^{(n)}\right]}{(\Delta x)^2}I+\frac{\tilde{\varepsilon}_{xx}^{(n)}}{(\Delta x)^2}S_{-M_x}\right\}\Phi_{\mathrm{e}}^{(n)}$$

$$+\left\{\frac{\tilde{\varepsilon}_{yy}^{(n+M_y)}}{(\Delta y)^2}S_{M_y}-\frac{\left[2A_{M_y}\tilde{\varepsilon}_{yy}^{(n)}\right]}{(\Delta y)^2}I+\frac{\tilde{\varepsilon}_{yy}^{(n)}}{(\Delta y)^2}S_{-M_y}\right\}\Phi_{\mathrm{e}}^{(n+)}$$

$$+\left\{\frac{\tilde{\varepsilon}_{zz}^{(n+M_z)}}{(\Delta z)^2}S_{M_z}-\frac{\left[2A_{M_z}\tilde{\varepsilon}_{zz}^{(n)}\right]}{(\Delta z)^2}I+\frac{\tilde{\varepsilon}_{zz}^{(n)}}{(\Delta z)^2}S_{-M_z}\right\}\Phi_{\mathrm{e}}^{(n)}$$

$$=\left\{\underbrace{\frac{\tilde{\varepsilon}_{zz}^{(n)}}{(\Delta z)^2}}_{=\alpha_{zz}^{(n)}}S_{-M_z}+\underbrace{\frac{\tilde{\varepsilon}_{yy}^{(n)}}{(\Delta y)^2}}_{=\alpha_{yy}^{(n)}}S_{-M_y}+\underbrace{\frac{\tilde{\varepsilon}_{xx}^{(n)}}{(\Delta x)^2}}_{=\alpha_{xx}^{(n)}}S_{-M_x}-\underbrace{\left\{\frac{\left[2A_{M_x}\tilde{\varepsilon}_{xx}^{(n)}\right]}{(\Delta x)^2}+\frac{\left[2A_{M_y}\tilde{\varepsilon}_{yy}^{(n)}\right]}{(\Delta y)^2}+\frac{\left[2A_{M_z}\tilde{\varepsilon}_{zz}^{(n)}\right]}{(\Delta z)^2}\right\}}_{=\alpha^{(n)}}I+\underbrace{\frac{\tilde{\varepsilon}_{xx}^{(n+M_x)}}{(\Delta x)^2}}_{=\alpha_{xx}^{(n+M_x)}}S_{M_x}+\underbrace{\frac{\tilde{\varepsilon}_{yy}^{(n+M_y)}}{(\Delta y)^2}}_{=\alpha_{yy}^{(n+M_y)}}S_{M_y}+\underbrace{\frac{\tilde{\varepsilon}_{zz}^{(n+M_z)}}{(\Delta z)^2}}_{=\alpha_{zz}^{(n+M_z)}}S_{M_z}\right\}\Phi_{\mathrm{e}}^{(n)}$$

$$=\left\{\alpha_{zz}^{(n)}S_{-M_z}+\alpha_{yy}^{(n)}S_{-M_y}+\alpha_{xx}^{(n)}S_{-M_x}-\alpha^{(n)}I+\alpha_{xx}^{(n+M_x)}S_{M_x}+\alpha_{yy}^{(n+M_y)}S_{M_y}+\alpha_{zz}^{(n+M_z)}S_{M_z}\right\}\Phi_{\mathrm{e}}^{(n)}$$

3-D FIT – Electrostatic Case / 3D-FIT – Elektrostatischer Fall

$$\oiint_{S=\partial V}\underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}}=\oiint_{S=\partial V}\left[\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\right]\bullet\underline{\mathbf{dS}}$$
$$=\iiint_V\rho_{\mathrm{e}}(\underline{\mathbf{R}},t)\,dV$$

$$\oiint_{S=\partial V}\underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}}=\oiint_{S=\partial V}\left[\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}})\bullet\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\right]\bullet\underline{\mathbf{dS}}$$
$$=-\left\{\frac{1}{(\Delta x)^2}\left(S_{M_x}-I\right)\tilde{\varepsilon}_{xx}^{(n)}(I-S_{-M_x})\,\Phi_{\mathrm{e}}^{(n)}+\frac{1}{(\Delta z)^2}\left(S_{M_z}-I\right)\tilde{\varepsilon}_{zz}^{(n)}(I-S_{-M_z})\,\Phi_{\mathrm{e}}^{(n)}+\frac{1}{(\Delta y)^2}\left(S_{M_y}-I\right)\tilde{\varepsilon}_{yy}^{(n)}(I-S_{-M_y})\right\}\Phi_{\mathrm{e}}^{(n)}$$
$$=-\left\{\alpha_{zz}^{(n)}S_{-M_z}+\alpha_{yy}^{(n)}S_{-M_y}+\alpha_{xx}^{(n)}S_{-M_x}-\alpha^{(n)}I+\alpha_{xx}^{(n+M_x)}S_{M_x}+\alpha_{yy}^{(n+M_y)}S_{M_y}+\alpha_{zz}^{(n+M_z)}S_{M_z}\right\}\Phi_{\mathrm{e}}^{(n)}$$

$$\iiint_V\rho_{\mathrm{e}}(\underline{\mathbf{R}},t)\,dV=\rho_{\mathrm{e}}^{(n)}(t)\Delta x\Delta y\Delta z$$

$$-\left(\alpha_{zz}^{(n)}S_{-M_z}+\alpha_{yy}^{(n)}S_{-M_y}+\alpha_{xx}^{(n)}S_{-M_x}-\alpha^{(n)}I+\alpha_{xx}^{(n+M_x)}S_{M_x}+\alpha_{yy}^{(n+M_y)}S_{M_y}+\alpha_{zz}^{(n+M_z)}S_{M_z}\right)\Phi_{\mathrm{e}}^{(n)}=\rho_{\mathrm{e}}^{(n)}(t)\Delta x\Delta y\Delta z$$

$$\left(-\alpha_{zz}^{(n)}S_{-M_z}-\alpha_{yy}^{(n)}S_{-M_y}-\alpha_{xx}^{(n)}S_{-M_x}+\alpha^{(n)}I-\alpha_{xx}^{(n+M_x)}S_{M_x}-\alpha_{yy}^{(n+M_y)}S_{M_y}-\alpha_{zz}^{(n+M_z)}S_{M_z}\right)\Phi_{\mathrm{e}}^{(n)}=\rho_{\mathrm{e}}^{(n)}(t)\Delta x\Delta y\Delta z$$

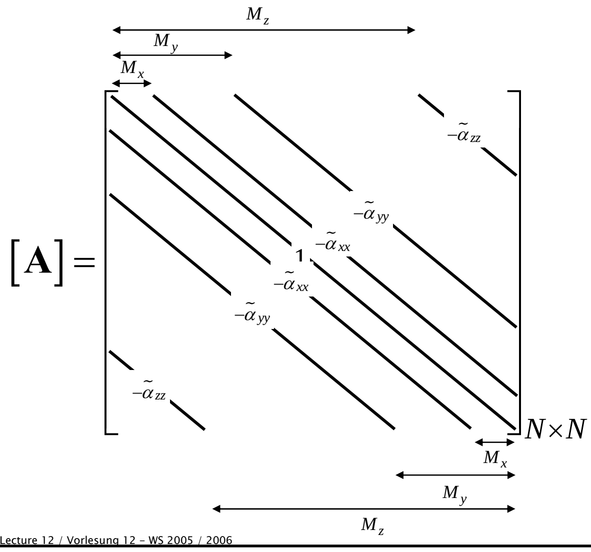
$$\left(-\underbrace{\frac{\alpha_{zz}^{(n)}}{\alpha^{(n)}}}_{=\tilde{\alpha}_{zz}^{(n)}}S_{-M_z}-\underbrace{\frac{\alpha_{yy}^{(n)}}{\alpha^{(n)}}}_{=\tilde{\alpha}_{yy}^{(n)}}S_{-M_y}-\underbrace{\frac{\alpha_{xx}^{(n)}}{\alpha^{(n)}}}_{=\tilde{\alpha}_{xx}^{(n)}}S_{-M_x}+I-\underbrace{\frac{\alpha_{xx}^{(n+M_x)}}{\alpha^{(n)}}}_{=\tilde{\alpha}_{xx}^{(n+M_x)}}S_{M_x}-\underbrace{\frac{\alpha_{yy}^{(n+M_y)}}{\alpha^{(n)}}}_{=\tilde{\alpha}_{yy}^{(n+M_y)}}S_{M_y}-\underbrace{\frac{\alpha_{zz}^{(n+M_z)}}{\alpha^{(n)}}}_{=\tilde{\alpha}_{zz}^{(n+M_z)}}S_{M_z}\right)\Phi_{\mathrm{e}}^{(n)}=\frac{\rho_{\mathrm{e}}^{(n)}(t)\Delta x\Delta y\Delta z}{\alpha^{(n)}}$$

$$\left(-\tilde{\alpha}_{zz}^{(n)}S_{-M_z}-\tilde{\alpha}_{yy}^{(n)}S_{-M_y}-\tilde{\alpha}_{xx}^{(n)}S_{-M_x}+I-\tilde{\alpha}_{xx}^{(n+M_x)}S_{M_x}-\tilde{\alpha}_{yy}^{(n+M_y)}S_{M_y}-\tilde{\alpha}_{zz}^{(n+M_z)}S_{M_z}\right)\Phi_{\mathrm{e}}^{(n)}=\frac{\rho_{\mathrm{e}}^{(n)}(t)\Delta x\Delta y\Delta z}{\alpha^{(n)}}$$

Discrete Poisson's Grid Equation / Diskrete Poissonsche Gittergleichung

$$\left(-\tilde{\alpha}_{zz}^{(n)} S_{-M_z} - \tilde{\alpha}_{yy}^{(n)} S_{-M_y} - \tilde{\alpha}_{xx}^{(n)} S_{-M_x} + I - \tilde{\alpha}_{xx}^{(n+M_x)} S_{M_x} - \tilde{\alpha}_{yy}^{(n+M_y)} S_{M_y} - \tilde{\alpha}_{zz}^{(n+M_z)} S_{M_z}\right)\Phi_{\mathrm{e}}^{(n)} = \frac{\rho_{\mathrm{e}}^{(n)}(t)\Delta x \Delta y \Delta z}{\alpha^{(n)}}$$

$$[\mathbf{A}]\{\mathbf{x}\} = \{\mathbf{b}\}$$



3-D FIT - Electrostatic Case / 3D-FIT - Elektrostatischer Fall

Electrostatic Poisson's grid equation / Elektrostatische Poissonsche Gittergleichung

$$[\widetilde{\mathbf{div}}][\tilde{\boldsymbol{\varepsilon}}][\tilde{\mathbf{S}}][\mathbf{R}]^{-1}[\mathbf{grad}]\{\Phi_{\mathrm{e}}\} = -[\tilde{\mathbf{V}}]\{\boldsymbol{\rho}_{\mathrm{e}}\}$$

Homogeneous isotropic case for a cubic grid complex /
Homogener isotroper Fall für ein kubischen Gitterkomplex

$$[\tilde{\boldsymbol{\varepsilon}}]_{3N\times 3N} = \varepsilon_0 \varepsilon_{\mathrm{r}} [\mathbf{I}]_{3N\times 3N}$$

$$[\tilde{\mathbf{S}}] = (\Delta x)^2 [\mathbf{I}]_{3N\times 3N}$$

$$[\mathbf{R}]^{-1} = \frac{1}{\Delta x}[\mathbf{I}]_{3N\times 3N}$$

$$[\tilde{\mathbf{V}}] = (\Delta x)^3 [\mathbf{I}]_{N\times N}$$

$$[\widetilde{\mathbf{div}}]\varepsilon_0\varepsilon_{\mathrm{r}}[\mathbf{I}](\Delta x)^2[\mathbf{I}]\frac{1}{\Delta x}[\mathbf{I}][\mathbf{grad}]\{\Phi_{\mathrm{e}}\} = -(\Delta x)^3[\mathbf{I}]\{\boldsymbol{\rho}_{\mathrm{e}}\}$$

$$[\widetilde{\mathbf{div}}][\mathbf{grad}]\{\Phi_{\mathrm{e}}\} = -\frac{(\Delta x)^2}{\varepsilon_0\varepsilon_{\mathrm{r}}}\{\boldsymbol{\rho}_{\mathrm{e}}\}$$

Electrostatic Poisson's grid equation / Elektrostatische Poissonsche Gittergleichung

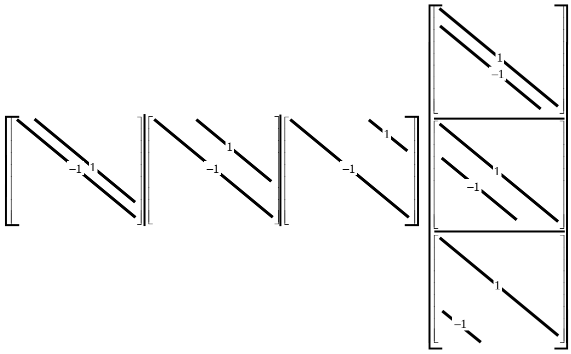
$$-[\widetilde{\mathbf{div}}][\mathbf{grad}]\{\Phi_{\mathrm{e}}\} = \frac{(\Delta x)^2}{\varepsilon_0\varepsilon_{\mathrm{r}}}\{\boldsymbol{\rho}_{\mathrm{e}}\}$$

Band Structure of the Div-Grad-Operator in Matrix Form / Bandstruktur des Div-Grad-Operators in Matrixform

$$
[\widetilde{\mathbf{div}}][\mathbf{grad}] = \left[[\mathbf{P}_x],[\mathbf{P}_y],[\mathbf{P}_z]\right]\begin{bmatrix} -[\mathbf{P}_x]^{\mathrm{T}} \\ -[\mathbf{P}_y]^{\mathrm{T}} \\ -[\mathbf{P}_z]^{\mathrm{T}} \end{bmatrix}
$$

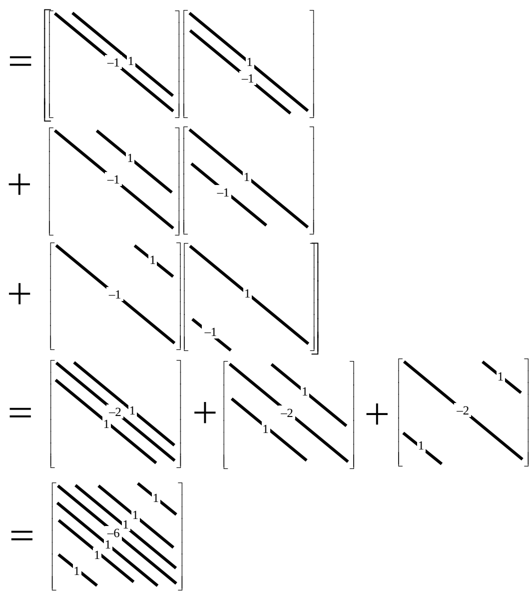
$$
= -\left[[\mathbf{P}_x][\mathbf{P}_x]^{\mathrm{T}} + [\mathbf{P}_y][\mathbf{P}_y]^{\mathrm{T}} + [\mathbf{P}_z][\mathbf{P}_z]^{\mathrm{T}}\right]_{N\times N}
$$

$[\widetilde{\mathbf{div}}][\mathbf{grad}] =$



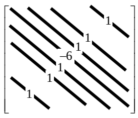
Band Structure of the Div-Grad-Operator in Matrix Form / Bandstruktur des Div-Grad-Operators in Matrixform

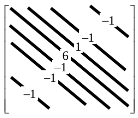
$[\widetilde{\mathbf{div}}][\mathbf{grad}] =$



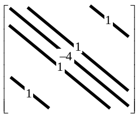
Band Structure of the Div-Grad-Operator in Matrix Form / Bandstruktur des Div-Grad-Operators in Matrixform

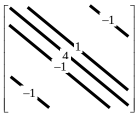
3-D case / 3D-Fall

$$\left[\widetilde{\mathbf{div}}\right]\left[\mathbf{grad}\right] =$$ 

$$-\left[\widetilde{\mathbf{div}}\right]\left[\mathbf{grad}\right] =$$ 

2-D case in the *xz* plane / 2D-Fall in der *xz*-Ebene

$$\left[\widetilde{\mathbf{div}}\right]\left[\mathbf{grad}\right] =$$ 

$$-\left[\widetilde{\mathbf{div}}\right]\left[\mathbf{grad}\right] =$$ 

Numerical Methods for Linear Systems / Numerische Methoden für lineare Gleichungssysteme

One the simplest type of iterative methods for solving the linear system /
Einer der einfachsten Typen von iterativen Methoden zur Lösung des linearen Gleichungssystems

$$\left[\mathbf{A}\right]\left\{\mathbf{x}\right\} = \left\{\mathbf{b}\right\}$$

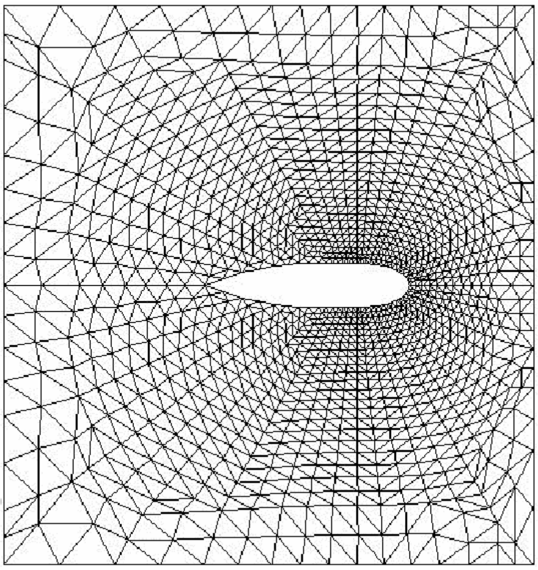
- Gaussian elimination (Gauss method) / Gaußsches Eliminationsverfahren (Gauß-Methode)

- Jacobi method (J method) / Jacobi-Methode (J-Methode)
- Gauss-Seidel method (GS method) / Gauß-Seidel-Methode (GS-Methode)
- Successive overrelaxation method (SOR method) / Überrelaxationsverfahren (SOR-Methode)
- Symmetric successive overrelaxation method (SSOR method) / Symmetrisches Überrelaxationsverfahren (SSOR-Methode)

- Conjugate gradient method (CG method) / Konjugierte Gradientenmethode (KG-Methode)
- Multi grid methods (MG method) / Mehrgitterverfahren (MG-Methode)
- Algebraic multi grid method (AMG method) / Algebraische Mehrgitterverfahren (AMG-Methode)

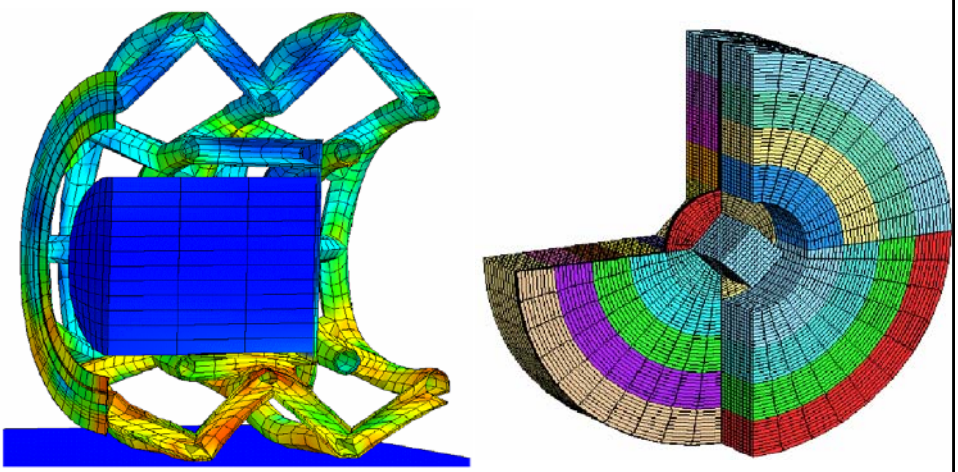
Numerical Methods for Linear Systems / Numerische Methoden für lineare Gleichungssysteme

Algebraic multi grid method (AMG method) / Algebraische Mehrgitterverfahren (AMG-Methode)



Numerical Methods for Linear Systems / Numerische Methoden für lineare Gleichungssysteme

Algebraic multi grid method (AMG method) / Algebraische Mehrgitterverfahren (AMG-Methode)



Iterative Methods for Linear Systems / Iterative Methoden für lineare Gleichungssysteme

$$[\mathbf{A}]\{\mathbf{x}\}=\{\mathbf{b}\}$$

LU decomposition of matrix [A] / LU-Zerlegung der Matrix [A]

$$[\mathbf{A}]=[\mathbf{L}]+[\mathbf{D}]+[\mathbf{U}]$$

Lower triangular matrix / Unter Dreiecksmatrix

$$[\mathbf{L}]=\begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 \\ A_{21} & 0 & \cdots & & \vdots \\ A_{31} & A_{32} & \ddots & & 0 \\ \vdots & & & 0 & 0 \\ A_{N1} & A_{N2} & \cdots & A_{N(N-1)} & 0 \end{bmatrix}_{N\times N}$$

Main diagonal matrix / Hauptdiagonalmatrix

$$[\mathbf{D}]=\left[\operatorname{diag}\{A_{11},A_{22},\cdots,A_{NN}\}\right]_{N\times N}$$

Upper triangular matrix / Obere Dreiecksmatrix

$$[\mathbf{U}]=\begin{bmatrix} 0 & A_{12} & \cdots & \cdots & A_{1N} \\ 0 & 0 & \cdots & & \vdots \\ 0 & 0 & \ddots & & A_{(N-2)N} \\ \vdots & & & 0 & A_{(N-1)N} \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}_{N\times N}$$

$$[\mathbf{A}]\{\mathbf{x}\}=\{\mathbf{b}\} \;\rightarrow\; \{[\mathbf{L}]+[\mathbf{D}]+[\mathbf{U}]\}\{\mathbf{x}\}=\{\mathbf{b}\} \;\rightarrow\; [\mathbf{D}]\{\mathbf{x}\}=-\{[\mathbf{L}]+[\mathbf{U}]\}\{\mathbf{x}\}+\{\mathbf{b}\}$$

Iterative Methods for Linear Systems / Iterative Methoden für lineare Gleichungssysteme

$$\begin{aligned}
[\mathbf{A}]\{\mathbf{x}\}&=\{\mathbf{b}\}\\
\{[\mathbf{L}]+[\mathbf{D}]+[\mathbf{U}]\}\{\mathbf{x}\}&=\{\mathbf{b}\}\\
[\mathbf{D}]\{\mathbf{x}\}&=-\{[\mathbf{L}]+[\mathbf{U}]\}\{\mathbf{x}\}+\{\mathbf{b}\}\\
\{\mathbf{x}\}&=\underbrace{-[\mathbf{D}]^{-1}\{[\mathbf{L}]+[\mathbf{U}]\}}_{=[\mathbf{G}]}\{\mathbf{x}\}+\underbrace{[\mathbf{D}]^{-1}\{\mathbf{b}\}}_{=\{\mathbf{c}\}}\\
&=[\mathbf{G}]\{\mathbf{x}\}+\{\mathbf{c}\}
\end{aligned}$$

$$\{\mathbf{x}\}=[\mathbf{G}]\{\mathbf{x}\}+\{\mathbf{c}\}$$

$$\{\mathbf{x}\}^{(l+1)}=[\mathbf{G}]\{\mathbf{x}\}^{(l)}+\{\mathbf{c}\} \quad l=1,2,\ldots,L$$

- Jacobi method (J method) / Jacobi-Methode (J-Methode)
- Gauss-Seidel method (GS method) / Gauß-Seidel-Methode (GS-Methode)
- Successive overrelaxation method (SOR method) / Überrelaxationsverfahren (SOR-Methode)
- Symmetric successive overrelaxation method (SSOR method) / Symmetrisches Überrelaxationsverfahren (SSOR-Methode)

Jacobi Method / Jacobi-Methode

$$\{\mathbf{x}\}^{(l+1)} = [\mathbf{G}]_\mathrm{J}\,\{\mathbf{x}\}^{(l)} + \{\mathbf{c}\}_\mathrm{J} \qquad l = 1, 2, \ldots, L$$

$$[\mathbf{G}]_\mathrm{J} = -[\mathbf{D}]^{-1}\{[\mathbf{L}] + [\mathbf{U}]\}$$

$$\{\mathbf{c}\}_\mathrm{J} = [\mathbf{D}]^{-1}\{\mathbf{b}\}$$

$$x_i^{(l+1)} = \sum_{j=1}^{N} G_{\mathrm{J},ij}\, x_j^{(l)} + c_{\mathrm{J},i} \qquad l = 1, 2, \ldots, L$$

It follows with the LU decomposition of matrix $[\mathbf{A}]$ /
Mit der LU-Zerlegung der Matrix $[\mathbf{A}]$ folgt

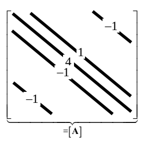
$$x_i^{(l+1)} = \sum_{j=1}^{i-1} G_{\mathrm{J},ij}\, x_j^{(l)} + \sum_{j=i+1}^{N} G_{\mathrm{J},ij}\, x_j^{(l)} + c_{\mathrm{J},i} \qquad l = 1, 2, \ldots, L$$

Jacobi Method / Jacobi-Methode

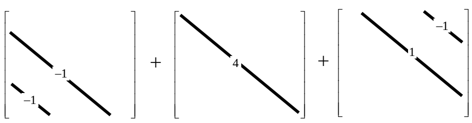
2-D case in the xz plane /
2D-Fall in der xz-Ebene

$$-[\widetilde{\mathbf{div}}][\mathbf{grad}]\{\Phi_\mathrm{e}\} = \frac{(\Delta x)^2}{\varepsilon_0 \varepsilon_\mathrm{r}}\{\boldsymbol{\rho}_\mathrm{e}\}$$

$-[\widetilde{\mathbf{div}}][\mathbf{grad}] =$

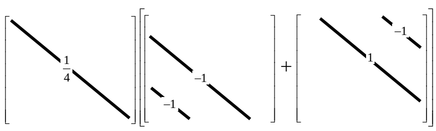


$[\mathbf{A}] = \{[\mathbf{L}] + [\mathbf{D}] + [\mathbf{U}]\} =$



$$\{\mathbf{x}\} = \underbrace{-[\mathbf{D}]^{-1}\{[\mathbf{L}] + [\mathbf{U}]\}}_{=[\mathbf{G}]_\mathrm{J}}\{\mathbf{x}\} + \underbrace{[\mathbf{D}]^{-1}\{\mathbf{b}\}}_{=\{\mathbf{c}\}_\mathrm{J}} = [\mathbf{G}]_\mathrm{J}\{\mathbf{x}\} + \{\mathbf{c}\}_\mathrm{J}$$

$[\mathbf{G}]_\mathrm{J} = -[\mathbf{D}]^{-1}\{[\mathbf{L}] + [\mathbf{U}]\} = -$



Jacobi & Gauss-Seidel Method / Jacobi- & Gauss-Seidel-Methode

$$\{\mathbf{x}\}^{(l+1)} = [\mathbf{G}]_{\mathrm{J}}\{\mathbf{x}\}^{(l)} + \{\mathbf{c}\}_{\mathrm{J}} \quad l = 1, 2, \ldots, L$$

$$[\mathbf{G}]_{\mathrm{J}} = -[\mathbf{D}]^{-1}\{[\mathbf{L}] + [\mathbf{U}]\}$$
$$\{\mathbf{c}\}_{\mathrm{J}} = [\mathbf{D}]^{-1}\{\mathbf{b}\}$$

$$x^{(n,l+1)} = \frac{1}{4}\left[x^{(n-M_z,l)} + x^{(n-M_x,l)} + x^{(n+M_x,l)} + x^{(n+M_z,l)} + b^{(n)}\right]$$

$$\{\mathbf{x}\}^{(l+1)} = [\mathbf{G}]_{\mathrm{GS}}\{\mathbf{x}\}^{(l)} + \{\mathbf{c}\}_{\mathrm{GS}} \qquad l = 1, 2, \ldots, L$$

$$[\mathbf{G}]_{\mathrm{GS}} = -([\mathbf{D}] + [\mathbf{L}])^{-1}[\mathbf{U}]$$
$$\{\mathbf{c}\}_{\mathrm{GS}} = ([\mathbf{D}] + [\mathbf{L}])^{-1}\{\mathbf{b}\}$$

$$x^{(n,l+1)} = \frac{1}{4}\left[x^{(n-M_z,l+1)} + x^{(n-M_x,l+1)} + x^{(n+M_x,l)} + x^{(n+M_z,l)} + b^{(n)}\right]$$

Successive Overrelaxation Method (SOR Method) / Überrelaxationsverfahren (SOR-Methode)

$$\{\mathbf{x}\}^{(l+1)} = \{\mathbf{x}\}^{(l)} - \omega\left[\overline{\{\mathbf{x}\}}^{(l+1)} - \{\mathbf{x}\}^{(l)}\right] \qquad l = 1, 2, \ldots, L$$
$$= (1-\omega)\{\mathbf{x}\}^{(l)} + \omega\overline{\{\mathbf{x}\}}^{(l+1)}$$

$\{\mathbf{x}\}^{(l+1)}$: Algebraic field vector at the iteration step l / Algebrischer Feldvektor zum Iterationsschritt l

$\overline{\{\mathbf{x}\}}^{(l+1)}$: Gauss-Seidel value at the iteration step l / Gauß-Seidel-Wert zum Iterationsschritt l

ω : Relaxatios Parameter / Relaxationsparameter

$0 < \omega < 1$: Under relaxation method / Unterrelaxationsmethode

$\omega = 1$: Gauss-Seidel method / Gauß-Seidel-Methode

$1 < \omega < 2$: Over relaxation method / Überrelaxationsmethod

Successive Overrelaxation Method (SOR Method) / Überrelaxationsverfahren (SOR-Methode)

$$\{\mathbf{x}\}^{(l+1)} = \{\mathbf{x}\}^{(l)} - \omega\left[\overline{\{\mathbf{x}\}}^{(l+1)} - \{\mathbf{x}\}^{(l)}\right] \qquad l = 1,2,\ldots,L$$
$$= (1-\omega)\{\mathbf{x}\}^{(l)} + \omega\overline{\{\mathbf{x}\}}^{(l+1)}$$

$$x_i^{(l+1)} = (1-\omega)\, x_i^{(l)} + \omega x_i^{(l+1)}$$
$$= (1-\omega)\, x_i^{(l)} + \omega\left\{\sum_{j=1}^{i-1} G_{\mathrm{J},ij}\; x_j^{(l+1)} + \sum_{j=i+1}^{N} G_{\mathrm{J},ij}\, x_j^{(l)} + c_{\mathrm{J},i}\right\} \qquad i = 1,2,\ldots,N$$

$$x_i^{(l+1)} = (1-\omega)\, x_i^{(l)} + \omega\left\{-\sum_{j=1}^{i-1} D_{ii}^{-1} L_{ij}\; x_j^{(l+1)} - \sum_{j=i+1}^{N} D_{ii}^{-1} U_{ij}\, x_j^{(l)} + c_{\mathrm{J},i}\right\} \qquad i = 1,2,\ldots,N$$

$$\{\mathbf{x}\}^{(l+1)} = (1-\omega)\{\mathbf{x}\}^{(l)} - \omega\left\{[\mathbf{D}]^{-1}[\mathbf{L}]\{\mathbf{x}\}^{(l+1)} + [\mathbf{D}]^{-1}[\mathbf{U}]\{\mathbf{x}\}^{(l)} - [\mathbf{D}]^{-1}\{\mathbf{b}\}\right\} \qquad i = 1,2,\ldots,N$$

$$\{\mathbf{x}\}^{(l+1)} = \underbrace{\left([\mathbf{D}] + \omega[\mathbf{L}]\right)^{-1}\left[(1-\omega)[\mathbf{D}] - \omega[\mathbf{U}]\right]}_{=[\mathbf{G}]_{\mathrm{SOR}}}\{\mathbf{x}\}^{(l)} + \underbrace{\omega\left([\mathbf{D}] + \omega[\mathbf{L}]\right)^{-1}\{\mathbf{b}\}}_{=\{\mathbf{c}\}_{\mathrm{SOR}}} \qquad i = 1,2,\ldots,N$$

Successive Overrelaxation Method (SOR Method) / Überrelaxationsverfahren (SOR-Methode)

$$\{\mathbf{x}\}^{(l+1)} = \{\mathbf{x}\}^{(l)} - \omega\left[\overline{\{\mathbf{x}\}}^{(l+1)} - \{\mathbf{x}\}^{(l)}\right] \qquad l = 1,2,\ldots,L$$
$$= (1-\omega)\{\mathbf{x}\}^{(l)} + \omega\overline{\{\mathbf{x}\}}^{(l+1)}$$

$$\{\mathbf{x}\}^{(l+1)} = \underbrace{\left([\mathbf{D}] + \omega[\mathbf{L}]\right)^{-1}\left[(1-\omega)[\mathbf{D}] - \omega[\mathbf{U}]\right]}_{=[\mathbf{G}]_{\mathrm{SOR}}}\{\mathbf{x}\}^{(l)} + \underbrace{\omega\left([\mathbf{D}] + \omega[\mathbf{L}]\right)^{-1}\{\mathbf{b}\}}_{=\{\mathbf{c}\}_{\mathrm{SOR}}} \qquad i = 1,2,\ldots,N$$

$$[\mathbf{G}]_{\mathrm{SOR}} = \left([\mathbf{D}] + \omega[\mathbf{L}]\right)^{-1}\left[(1-\omega)[\mathbf{D}] - \omega[\mathbf{U}]\right]$$
$$\{\mathbf{c}\}_{\mathrm{SOR}} = \omega\left([\mathbf{D}] + \omega[\mathbf{L}]\right)^{-1}\{\mathbf{b}\}$$

$$\{\mathbf{x}\}^{(l+1)} = [\mathbf{G}]_{\mathrm{SOR}}\{\mathbf{x}\}^{(l)} + \{\mathbf{c}\}_{\mathrm{SOR}} \qquad l = 1,2,\ldots,L$$

Symmetric Successive Overrelaxation Method (SSOR Method) / Symmetrische Überrelaxationsverfahren (SSOR-Methode)

Forward SOR step / Vorwärts-SOR-Schritt

$$x_i^{\left(l+\frac{1}{2}\right)} = (1-\omega)\, x_i^{(l)} + \omega \left\{ \sum_{j=1}^{i-1} G_{\mathrm{J},ij}\, x_j^{\left(l+\frac{1}{2}\right)} + \sum_{j=i+1}^{N} G_{\mathrm{J},ij}\, x_j^{(l)} + c_{\mathrm{J},i} \right\} \qquad i = 1,2,\ldots,N$$

Backward SOR step / Rückwärts-SOR-Schritt

$$x_i^{(l+1)} = (1-\omega)\, x_i^{\left(l+\frac{1}{2}\right)} + \omega \left\{ \sum_{j=1}^{i-1} G_{\mathrm{J},ij}\, x_j^{\left(l+\frac{1}{2}\right)} + \sum_{j=i+1}^{N} G_{\mathrm{J},ij}\, x_j^{(l+1)} + c_{\mathrm{J},i} \right\} \qquad i = N, N-1,\ldots,1$$

Forward SOR step / Vorwärts-SOR-Schritt

$$\{\mathbf{x}\}^{(l+\frac{1}{2})} = (1-\omega)\{\mathbf{x}\}^{(l)} - \omega \left\{ [\mathbf{D}]^{-1}[\mathbf{L}]\{\mathbf{x}\}^{(l+\frac{1}{2})} + [\mathbf{D}]^{-1}[\mathbf{U}]\{\mathbf{x}\}^{(l)} - [\mathbf{D}]^{-1}\{\mathbf{b}\} \right\} \qquad i = 1,2,\ldots,N$$

Backward SOR step / Rückwärts-SOR-Schritt

$$\{\mathbf{x}\}^{(l+1)} = (1-\omega)\{\mathbf{x}\}^{(l+\frac{1}{2})} - \omega \left\{ [\mathbf{D}]^{-1}[\mathbf{L}]\{\mathbf{x}\}^{(l+\frac{1}{2})} + [\mathbf{D}]^{-1}[\mathbf{U}]\{\mathbf{x}\}^{(l)} - [\mathbf{D}]^{-1}\{\mathbf{b}\} \right\} \qquad i = N, N-1,\ldots,1$$

Convergence of Point Iterative Methods / Konvergenz von punktiterativen Methoden

$$\rho([\mathbf{G}]) = \max_{n=1,2,\ldots,N} \left| \nu_n([\mathbf{G}]) \right|$$

$$\rho([\mathbf{G}]) < 1$$

$$0 < \omega_{\mathrm{SOR}} < 2$$

$$0 < \omega_{\mathrm{SSOR}} < 2$$

Error Vector and Error Measure / Fehlervektor und Fehlermaß

$$\omega_{\mathrm{SOR,opt}} = \frac{2}{1+\sqrt{1-\rho\left([\mathbf{G}]_{\mathrm{J}}\right)}} = 1 + \left(\frac{\rho\left([\mathbf{G}]_{\mathrm{J}}\right)}{1+\sqrt{1-\rho\left([\mathbf{G}]_{\mathrm{J}}\right)}}\right)^2$$

$$\rho\left([\mathbf{G}]_{\mathrm{SOR}}\right) = \left(\frac{\rho\left([\mathbf{G}]_{\mathrm{J}}\right)}{1+\sqrt{1-\rho\left([\mathbf{G}]_{\mathrm{J}}\right)}}\right)^2$$

Symmetric positive definite [A] matrix /
Symmetrische positiv-definite [A] Matrix

$$\rho\left([\mathbf{L}][\mathbf{U}]\right) \le \frac{1}{4}$$

$$\omega_{\mathrm{SSOR}} = \frac{2}{1+2\sqrt{1-\rho\left([\mathbf{G}]_{\mathrm{J}}\right)}}$$

Approximation for a D-dimensional Dirichlet problem /
Approximation für ein D-dimensionales Dirichlet-Problem

$$\rho\left([\mathbf{G}]_{\mathrm{J}}\right) = \frac{\sum_{d=1}^{D} \cos\left(\frac{\pi}{N_d}\right)}{D}$$

Optimal Relaxation / Optimaler Relaxationsparameter

$$\{\mathbf{x}\}^{(l+1)} = [\mathbf{G}]\{\mathbf{x}\}^{(l)} + \{\mathbf{c}\}$$

$$\{\mathbf{x}\} = [\mathbf{G}]\{\mathbf{x}\} + \{\mathbf{c}\}$$

$$\{\boldsymbol{\varepsilon}\}^{(l)} = \{\mathbf{x}\}^{(l)} + \{\mathbf{x}\}$$

$$\{\boldsymbol{\varepsilon}\}^{(l+1)} = [\mathbf{G}]\{\boldsymbol{\varepsilon}\}^{(l)}$$

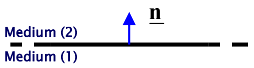
$$\lim_{l\to\infty} \{\boldsymbol{\varepsilon}\}^{(l)} = \{0\}$$

$$\varepsilon_{\mathrm{max}}^{(l+1)} = \max_{n=1,2,\dots,N} \left|\varepsilon^{(n,l+1)} - \varepsilon^{(n,l)}\right|$$

$$\varepsilon_{\mathrm{rel,max}}^{(l+1)} = \max_{n=1,2,\dots,N} \left|\frac{\varepsilon^{(n,l+1)} - \varepsilon^{(n,l)}}{\varepsilon^{(n,l+1)}}\right|$$

Electrostatic (ES) Fields / Elektrostatische (ES) Felder

Transition Conditions / Übergangsbedingungen



$\underline{\mathbf{R}} \in S_{\mathrm{I(nterface)}}$

$$E_{tan}^{(2)}\left(\underline{\mathbf{R}}\right) - E_{tan}^{(1)}\left(\underline{\mathbf{R}}\right) = 0$$

$$D_n^{(2)}\left(\underline{\mathbf{R}}\right) - D_n^{(2)}\left(\underline{\mathbf{R}}\right) = \eta_{\mathrm{e}}(\underline{\mathbf{R}})$$

$$E_{tan}^{(2)}\left(\underline{\mathbf{R}}\right) - E_{tan}^{(1)}\left(\underline{\mathbf{R}}\right) = 0$$

$$\Downarrow$$

$$\Phi_{\mathrm{e}}^{(2)}\left(\underline{\mathbf{R}}\right) - \Phi_{\mathrm{e}}^{(1)}\left(\underline{\mathbf{R}}\right) = \Phi_{\mathrm{e}0} = \mathrm{const.}$$

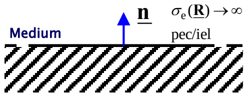
$$D_n^{(2)}\left(\underline{\mathbf{R}}\right) - D_n^{(2)}\left(\underline{\mathbf{R}}\right) = \eta_{\mathrm{e}}(\underline{\mathbf{R}})$$

$$\Downarrow$$

$$\frac{\partial}{\partial n}\Phi_{\mathrm{e}}^{(2)}\left(\underline{\mathbf{R}}\right) - \frac{\varepsilon_{\mathrm{r}}^{(1)}}{\varepsilon_{\mathrm{r}}^{(2)}}\frac{\partial}{\partial n}\Phi_{\mathrm{e}}^{(1)}\left(\underline{\mathbf{R}}\right) = -\frac{1}{\varepsilon_0\varepsilon_{\mathrm{r}}^{(2)}}\eta_{\mathrm{e}}(\underline{\mathbf{R}})$$

Boundary Conditions / Randbedingungen

$\underline{\mathbf{R}} \in S_{\mathrm{B(oundary)}}$



$\sigma_{\mathrm{e}}(\underline{\mathbf{R}}) \to \infty$

$$E_{tan}\left(\underline{\mathbf{R}}\right) = 0 \qquad \text{pec / iel}$$

$$D_n\left(\underline{\mathbf{R}}\right) = \eta_{\mathrm{e}}(\underline{\mathbf{R}}) \qquad \text{pec / iel}$$

$$E_{tan}\left(\underline{\mathbf{R}}\right) = 0$$

$$\Downarrow$$

$$\Phi_{\mathrm{e}}\left(\underline{\mathbf{R}}\right) = \Phi_{\mathrm{e}0} = \mathrm{const.}$$

$$D_n\left(\underline{\mathbf{R}}\right) = \eta_{\mathrm{e}}(\underline{\mathbf{R}})$$

$$\Downarrow$$

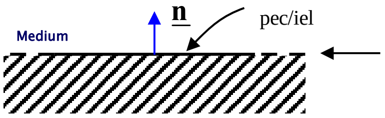
$$\frac{\partial}{\partial n}\Phi_{\mathrm{e}}\left(\underline{\mathbf{R}}\right) = -\frac{1}{\varepsilon_0\varepsilon_{\mathrm{r}}}\eta_{\mathrm{e}}(\underline{\mathbf{R}})$$

Electrostatic (ES) Fields / Elektrostatische (ES) Felder

Boundary conditions for the electrostatic potential / Randbedingungen für die elektrostatische Potential

$\underline{\mathbf{E}}(\underline{\mathbf{R}}), \underline{\mathbf{D}}(\underline{\mathbf{R}}), \Phi_{\mathrm{e}}(\underline{\mathbf{R}})$

$\underline{\mathbf{R}} \in S_{\mathrm{B(oundary)}}$



$\sigma_{\mathrm{e}}(\underline{\mathbf{R}}) \to \infty$

$\eta_{\mathrm{e}}(\underline{\mathbf{R}})$

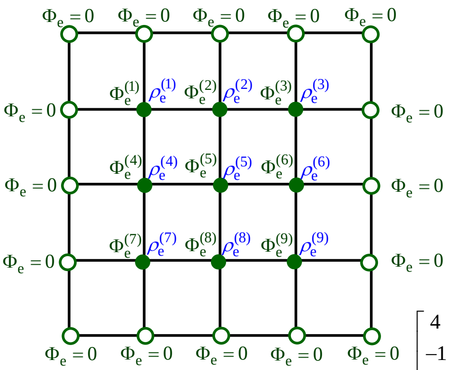
$$\Phi_{\mathrm{e}}\left(\underline{\mathbf{R}}\right) = \Phi_{\mathrm{e}0} = \mathrm{const.} \quad (\Phi_{\mathrm{e}0} = 0\ \mathrm{V})$$

$$\frac{\partial}{\partial n}\Phi_{\mathrm{e}}\left(\underline{\mathbf{R}}\right) = -\frac{1}{\varepsilon_0\varepsilon_{\mathrm{r}}}\eta_{\mathrm{e}}(\underline{\mathbf{R}})$$

Neumann Boundary Condition for Φ_{e} / Neumann-Randbedingung für Φ_{e}

Dirichlet Boundary Condition for Φ_{e} / Dirichlet-Randbedingung für Φ_{e}

Example: 2-D Dirichlet Problem / Beispiel: 2D-Dirichlet-Problem



$$-\underbrace{[\widetilde{\mathbf{div}}][\mathbf{grad}]}_{=[\mathbf{A}]}\underbrace{\{\Phi_e\}}_{=\{\mathbf{x}\}} = \underbrace{\frac{(\Delta x)^2}{\varepsilon_0\varepsilon_r}\{\boldsymbol{\rho}_e\}}_{=\{\mathbf{b}\}}$$

Dirichlet boundary condition for Φ_e /
Dirichlet-Randbedingung für Φ_e

$$x^{(n)} = \Phi_e^{(n)} = 0 \qquad n \in C_D = \partial S$$

$$\underline{\mathbf{R}} = x\underline{\mathbf{e}}_x + z\underline{\mathbf{e}}_z$$

$$-\Delta\Phi_e(\underline{\mathbf{R}}) = \frac{\rho_e(\underline{\mathbf{R}})}{\varepsilon_0\varepsilon_r} \qquad \underline{\mathbf{R}} \in S$$

$$-\oint\!\!\!\oint_{S=\partial V}\left[\nabla\Phi_e(\underline{\mathbf{R}})\right]\bullet \underline{\mathbf{dS}} = \frac{1}{\varepsilon_0\varepsilon_r}\iiint_V \rho_e(\underline{\mathbf{R}})\,\mathrm{d}V \qquad \underline{\mathbf{R}} \in S$$

Dirichlet boundary condition for Φ_e /
Dirichlet-Randbedingung für Φ_e

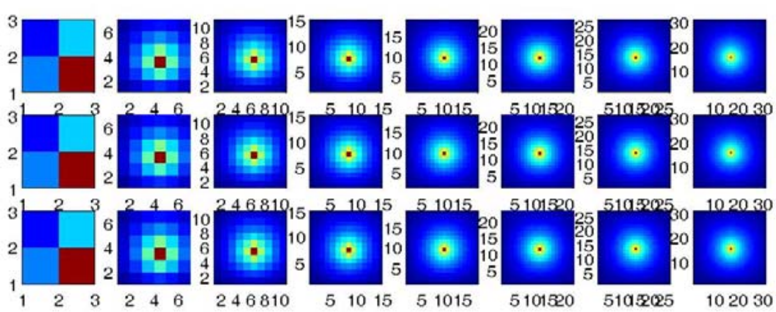
$$\Phi_e(\underline{\mathbf{R}}) = 0 \qquad \underline{\mathbf{R}} \in C_D = \partial S$$

$$\begin{bmatrix} 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 \end{bmatrix} \begin{Bmatrix} \Phi_e^{(1)} \\ \Phi_e^{(2)} \\ \Phi_e^{(3)} \\ \Phi_e^{(4)} \\ \Phi_e^{(5)} \\ \Phi_e^{(6)} \\ \Phi_e^{(7)} \\ \Phi_e^{(8)} \\ \Phi_e^{(9)} \end{Bmatrix} = \begin{Bmatrix} b^{(1)} \\ b^{(2)} \\ b^{(3)} \\ b^{(4)} \\ b^{(5)} \\ b^{(6)} \\ b^{(7)} \\ b^{(8)} \\ b^{(9)} \end{Bmatrix}$$

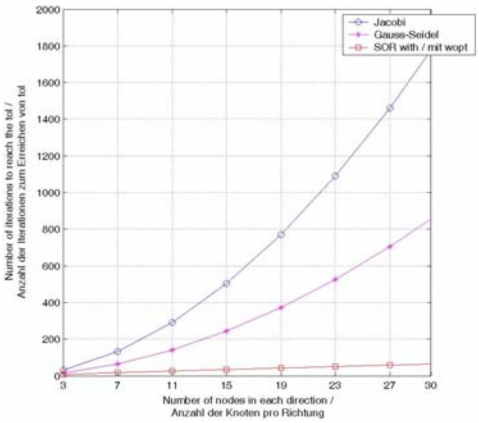
Example: 2-D Dirichlet Problem / Beispiel: 2D-Dirichlet-Problem

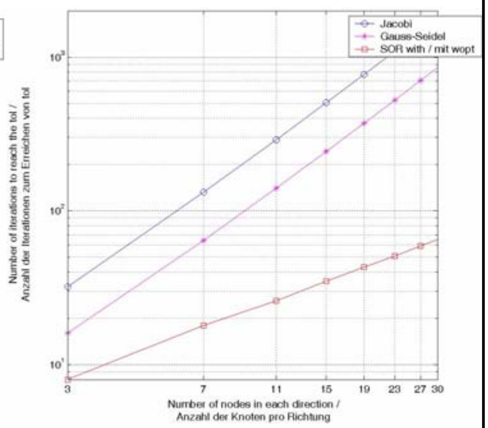
$$\rho_e(x,z) = \xi_{e0}\,\delta(x-x_s)\delta(z-z_s) \qquad \underline{\mathbf{R}} = \underline{\mathbf{R}}_s = x_s\underline{\mathbf{e}}_x + z_s\underline{\mathbf{e}}_z$$

$N = 3$ $N = 7$ $N = 11$ $N = 15$ $N = 19$ $N = 23$ $N = 27$ $N = 31$



Example: 2-D Dirichlet Problem / Beispiel: 2D-Dirichlet-Problem





End of Lecture 12 /
Ende der 12. Vorlesung