Numerical Methods of Electromagnetic Field Theory I (NFT I)
Numerische Methoden der Elektromagnetischen Feldtheorie I (NFT I) /

3rd Lecture / 3. Vorlesung

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Finite Difference (FD) Method / Finite Differenzen (FD) Methode
1-D FD Operators / 1D-FD-Operatoren

Backward FD Operator / Rückwärts-FD-Operator

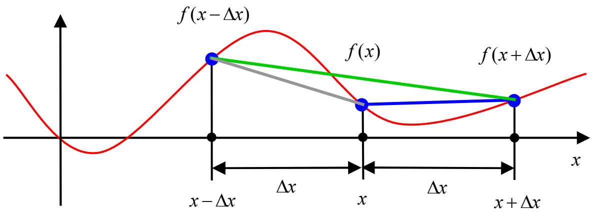
$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) = \frac{f(x) - f(x-\Delta x)}{\Delta x} + \mathrm{O}(\Delta x) \approx \frac{f(x) - f(x-\Delta x)}{\Delta x}$$

Forward FD Operator / Vorwärts-FD-Operator

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) = \frac{f(x+\Delta x) - f(x)}{\Delta x} + \mathrm{O}(\Delta x) \approx \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Central FD Operator / Zentraler FD-Operator

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) = \frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x} + \mathrm{O}[(\Delta x)^2] \approx \frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x}$$



FD Method – 1-D FD Operator of Second Order / FD-Methode – 1D-FD-Operator zweiter Ordnung

Derivative of the second order / Ableitung der zweiten Ordnung

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} f(x) \approx \frac{f(x+\Delta x) - 2f(x) + f(x-\Delta x)}{(\Delta x)^2} \quad (1)$$

Taylor series expansions / Taylor-Reihenentwicklungen

$$f(x+\Delta x) = f(x) + \Delta x \frac{\mathrm{d}f(x)}{\mathrm{d}x} + \frac{(\Delta x)^2}{2!}\frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2} + \frac{(\Delta x)^3}{3!}\frac{\mathrm{d}^3 f(x)}{\mathrm{d}x^3} + \frac{(\Delta x)^4}{4!}\frac{\mathrm{d}^4 f(x)}{\mathrm{d}x^4} + O\left[(\Delta x)^5\right] \quad (2)$$

$$f(x) = f(x) \quad (3)$$

$$f(x-\Delta x) = f(x) - \Delta x \frac{\mathrm{d}f(x)}{\mathrm{d}x} + \frac{(\Delta x)^2}{2!}\frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2} - \frac{(\Delta x)^3}{3!}\frac{\mathrm{d}^3 f(x)}{\mathrm{d}x^3} + \frac{(\Delta x)^4}{4!}\frac{\mathrm{d}^4 f(x)}{\mathrm{d}x^4} + O\left[(\Delta x)^5\right] \quad (4)$$

Multiply (2) with α, (3) with β, and (4) with γ /
Multipliziere (2) mit α, (3) mit β und (4) mit γ

$$\alpha f(x+\Delta x) = \alpha f(x) + \alpha\Delta x \frac{df(x)}{dx} + \alpha\frac{(\Delta x)^2}{2!}\frac{d^2 f(x)}{dx^2} + \alpha\frac{(\Delta x)^3}{3!}\frac{d^3 f(x)}{dx^3} + \alpha\frac{(\Delta x)^4}{4!}\frac{d^4 f(x)}{dx^4} + O\left[(\Delta x)^5\right] \quad (5)$$

$$\beta f(x) = \beta f(x) \quad (6)$$

$$\gamma f(x-\Delta x) = \gamma f(x) - \gamma\Delta x \frac{df(x)}{dx} + \gamma\frac{(\Delta x)^2}{2!}\frac{d^2 f(x)}{dx^2} - \gamma\frac{(\Delta x)^3}{3!}\frac{d^3 f(x)}{dx^3} + \gamma\frac{(\Delta x)^4}{4!}\frac{d^4 f(x)}{dx^4} + O\left[(\Delta x)^5\right] \quad (7)$$

FD Method – 1-D FD Operator of Second Order / FD-Methode – 1D-FD-Operator zweiter Ordnung

Add Equations (5)-(7) /
Addiere die Gleichungen (5)-(7)

$$\begin{aligned}
&\alpha f(x+\Delta x) + \beta f(x) + \gamma f(x-\Delta x) \\
&= (\alpha+\beta+\gamma) f(x) + (\alpha-\gamma)\Delta x \frac{df(x)}{dx} + (\alpha+\gamma)\frac{(\Delta x)^2}{2!}\frac{d^2 f(x)}{dx^2} \\
&+ (\alpha-\gamma)\frac{(\Delta x)^3}{3!}\frac{d^3 f(x)}{dx^3} + (\alpha+\gamma)\frac{(\Delta x)^4}{4!}\frac{d^4 f(x)}{dx^4} \\
&+ (\alpha-\gamma)\frac{(\Delta x)^5}{5!}\frac{d^5 f(x)}{dx^5} + (\alpha+\gamma)\frac{(\Delta x)^6}{6!}\frac{d^6 f(x)}{dx^6} + O\left[(\Delta x)^7\right]
\end{aligned}$$

$$\alpha - \gamma = 0 \quad \rightarrow \quad \gamma = \alpha$$

$$\alpha + \beta + \gamma = 0 \rightarrow \quad \beta = -2\alpha$$

FD Method – 1-D FD Operator of Second Order / FD-Methode – 1D-FD-Operator zweiter Ordnung

With the parameters / Mit den Parametern

$$\alpha=\gamma=1$$
$$\beta=-2$$

$$f(x+\Delta x)-2f(x)+f(x-\Delta x)=2\frac{(\Delta x)^2}{2}\frac{d^2f(x)}{dx^2}+2\frac{(\Delta x)^4}{\underbrace{4!}_{1\cdot2\cdot3\cdot4=24}}\frac{d^4f(x)}{dx^4}+2\frac{(\Delta x)^6}{\underbrace{6!}_{1\cdot2\cdot3\cdot4\cdot5\cdot6=720}}\frac{d^6f(x)}{dx^6}+O\left[(\Delta x)^8\right]$$

$$\frac{f(x+\Delta x)-2f(x)+f(x-\Delta x)}{(\Delta x)^2}=\frac{d^2f(x)}{dx^2}+\frac{(\Delta x)^2}{12}\frac{d^4f(x)}{dx^4}+\frac{(\Delta x)^6}{360}\frac{d^6f(x)}{dx^6}+O\left[(\Delta x)^8\right]$$

$$\frac{d^2f(x)}{dx^2}=\frac{f(x+\Delta x)-2f(x)+f(x-\Delta x)}{(\Delta x)^2}-\frac{(\Delta x)^2}{12}\frac{d^4f(x)}{dx^4}+\underbrace{\frac{(\Delta x)^4}{360}\frac{d^6f(x)}{dx^6}+O\left[(\Delta x)^8\right]}_{O[(\Delta x)^4]}$$

$$\frac{d^2f(x)}{dx^2}=\frac{f(x+\Delta x)-2f(x)+f(x-\Delta x)}{(\Delta x)^2}\underbrace{-\frac{(\Delta x)^2}{12}\frac{d^4f(x)}{dx^4}+\frac{(\Delta x)^4}{360}\frac{d^6f(x)}{dx^6}+O\left[(\Delta x)^8\right]}_{O[(\Delta x)^2]}$$

$$\frac{d^2f(x)}{dx^2}=\frac{f(x+\Delta x)-2f(x)+f(x-\Delta x)}{(\Delta x)^2}+O[(\Delta x)^2]$$

FD Method – 1-D FD Operator of Second Order / FD-Methode – 1D-FD-Operator zweiter Ordnung

$$\alpha f(x+\Delta x)+\beta f(x)+\gamma f(x-\Delta x)$$
$$=(\alpha+\beta+\gamma)f(x)+(\alpha-\gamma)\Delta x\frac{df(x)}{dx}+(\alpha+\gamma)\frac{(\Delta x)^2}{2!}\frac{d^2f(x)}{dx^2}$$
$$+(\alpha-\gamma)\frac{(\Delta x)^3}{3!}\frac{d^3f(x)}{dx^3}+(\alpha+\gamma)\frac{(\Delta x)^4}{4!}\frac{d^4f(x)}{dx^4}+O\left[(\Delta x)^5\right]$$

$$\alpha-\gamma=0 \quad\rightarrow\quad \gamma=\alpha$$
$$\alpha+\beta+\gamma=0\rightarrow\quad \beta=-2\alpha$$

With the parameters / Mit den Parametern

$$\alpha=\gamma=1$$
$$\beta=-2$$

$$f(x+\Delta x)-2f(x)+f(x-\Delta x)=2\frac{(\Delta x)^2}{2}\frac{d^2f(x)}{dx^2}+2\frac{(\Delta x)^4}{\underbrace{4!}_{1\cdot2\cdot3\cdot4=24}}\frac{d^4f(x)}{dx^4}+O\left[(\Delta x)^6\right]$$

$$\frac{f(x+\Delta x)-2f(x)+f(x-\Delta x)}{(\Delta x)^2}=\frac{d^2f(x)}{dx^2}+\frac{(\Delta x)^2}{12}\frac{d^4f(x)}{dx^4}+O\left[(\Delta x)^4\right]$$

$$\frac{d^2f(x)}{dx^2}=\frac{f(x+\Delta x)-2f(x)+f(x-\Delta x)}{(\Delta x)^2}\underbrace{-\frac{(\Delta x)^2}{12}\frac{d^4f(x)}{dx^4}+O\left[(\Delta x)^4\right]}_{O[(\Delta x)^2]}$$

$$\frac{d^2f(x)}{dx^2}=\frac{f(x+\Delta x)-2f(x)+f(x-\Delta x)}{(\Delta x)^2}+O[(\Delta x)^2]$$

FD Method – 1-D FD Operators of Second Order / FD-Methode – 1D-FD-Operatoren zweiter Ordnung

Function of one variable / Funktion einer Variablen

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} f(x) = \frac{f(x+\Delta x) - 2f(x) + f(x-\Delta x)}{(\Delta x)^2} + \bigcirc[(\Delta x)^2]$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} f(t) = \frac{f(t+\Delta t) - 2f(t) + f(t-\Delta t)}{(\Delta t)^2} + \bigcirc[(\Delta t)^2]$$

Function of two variables / Funktion von zwei Variablen

$$\frac{\partial^2}{\partial x^2} f(x,t) = \frac{f(x+\Delta x,t) - 2f(x,t) + f(x-\Delta x,t)}{(\Delta x)^2} + \bigcirc[(\Delta x)^2]$$

$$\frac{\partial^2}{\partial t^2} f(x,t) = \frac{f(x,t+\Delta t) - 2f(x,t) + f(x,t-\Delta t)}{(\Delta t)^2} + \bigcirc[(\Delta t)^2]$$

FD Method – 1-D Wave Equation / FD-Methode – 1D Wellengleichung

$$\frac{\partial^2}{\partial z^2} E_x(z,t) - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} E_x(z,t) = -\frac{\partial}{\partial z}\underbrace{J_{\mathrm{my}}(z,t)}_{=0} + \mu_0 \frac{\partial}{\partial t} J_{\mathrm{ex}}(z,t)$$

$$= \mu_0 \frac{\partial}{\partial t} J_{\mathrm{ex}}(z,t)$$

Central FD Operators / Zentrale FD-Operatoren

$$\frac{\partial^2}{\partial z^2} E_x(z,t) = \frac{E_x(z+\Delta z,t) - 2E_x(z,t) + E_x(z-\Delta z,t)}{(\Delta z)^2} + \bigcirc[(\Delta z)^2]$$

$$\frac{\partial^2}{\partial t^2} E_x(z,t) = \frac{E_x(z,t+\Delta t) - 2E_x(z,t) + E_x(z,t-\Delta t)}{(\Delta t)^2} + \bigcirc[(\Delta t)^2]$$

Backward FD Operator / Rückwärts-FD-Operator

$$\frac{\partial}{\partial t} J_{\mathrm{ex}}(z,t) = \frac{J_{\mathrm{ex}}(z,t) - J_{\mathrm{ex}}(z,t-\Delta t)}{\Delta t} + \bigcirc(\Delta t)$$

$$\frac{E_x(z+\Delta z,t) - 2E_x(z,t) + E_x(z-\Delta z,t)}{(\Delta z)^2} - \frac{1}{c_0^2}\frac{E_x(z,t+\Delta t) - 2E_x(z,t) + E_x(z,t-\Delta t)}{(\Delta t)^2} = \mu_0 \frac{J_{\mathrm{ex}}(z,t) - J_{\mathrm{ex}}(z,t-\Delta t)}{\Delta t}$$

$$+ \bigcirc[(\Delta z)^2] + \bigcirc[(\Delta t)^2]$$

FD Method – 1D Wave Equation / FD-Methode – 1D Wellengleichung

Explicit FD algorithm in the time domain of 2nd order in space and time /
Expliziter FD-Algorithmus im Zeitbereich 2ter Ordnung in Raum und Zeit

$$E_x(z,t+\Delta t) = 2E_x(z,t) - E_x(z,t-\Delta t) + c_0^2 \frac{(\Delta t)^2}{(\Delta z)^2}\left[E_x(z+\Delta z,t) - 2E_x(z,t) + E_x(z-\Delta z,t)\right]$$
$$+ c_0^2 \mu_0 \Delta t \left[J_{\mathrm{ex}}(z,t) - J_{\mathrm{ex}}(z,t-\Delta t)\right] + O[(\Delta z)^2] + O[(\Delta t)^2]$$

Marching-on-in-time algorithm /
„Marschieren in der Zeit"-Algorithmus

$$z \to n_z \Delta z, \quad n_z = 1,\ldots,N_z$$
$$t \to n_t \Delta t, \quad n_t = 1,\ldots,N_t$$

$$E_x(z,t) \to E_x^{(n_z,n_t)}$$
$$J_{\mathrm{ex}}(z,t) \to J_{\mathrm{ex}}^{(n_z,n_t)}$$

$$E_x^{(n_z,n_t+1)} = 2E_x^{(n_z,n_t)} - E_x^{(n_z,n_t-1)} + c_0^2 \frac{(\Delta t)^2}{(\Delta z)^2}\left[E_x^{(n_z+1,n_t)} - 2E_x^{(n_z,n_t)} + E_x^{(n_z-1,n_t)}\right] + c_0^2 \mu_0 \Delta t \left[J_{\mathrm{ex}}^{(n_z,n_t)} - J_{\mathrm{ex}}^{(n_z,n_t-1)}\right]$$

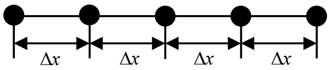
$$\Delta z = ?$$
$$\Delta t = ?$$

FD Method – Properties / FD-Methode – Eigenschaften

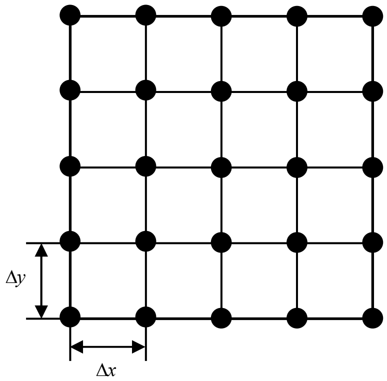
- Spatial and Temporal Discretization / Räumliche und zeitliche Diskretisierung $\boxed{\Delta z = ?,\ \Delta t = ?}$
- Consistency / Konsistenz
- Dissipation / Dissipation
- Stability Condition / Stabilitätsbedingung $\boxed{\Delta t = f(\Delta z)}$
- Convergence / Konvergenz

FD Method – 1-D, 2-D, 3-D Grid System / FD-Methode – 1D-, 2D- und 3D-Gittersystem

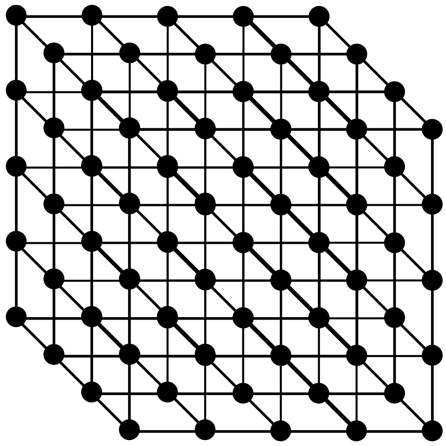
1-D Node-Based Grid /
1D knotenbasiertes Gitter



2-D Node-Based Grid /
2D knotenbasiertes Gitter



3-D Node-Based Grid /
3D knotenbasiertes Gitter



● Nodes with Assigned Field Quantities / Knoten mit zugeordneten Feldgrößen: Φ [V], $\underline{\mathbf{E}}$ [V/m], $\underline{\mathbf{H}}$ [A/m], $\underline{\mathbf{A}}$ [Vs/m]

FD Method – Grid Size / FD-Methode – Gittergröße

Sampling Theorem in Space / Abtastkriterium im Raum

$$\Delta x \leq \frac{\lambda_{\min}}{2}$$

Δx : **Spatial grid size / Räumliche Gittergröße**

$\lambda_{\min}$: **Minimal wavelength / Minimale Wellenlänge**

$$\lambda_{\min} = \frac{c_{\min}}{f_{\max}}$$

$c_{\min}$: **Minimal phase velocity / Minimale Phasengeschwindigkeit**

$f_{\max}$: **Maximal frequency / Maximale Frequenz**

Sampling Resolution / Abtastauflösung

$$G = \frac{\lambda_{\min}}{\Delta x} \quad G = 10, \ldots, 30$$

$$\Delta x = \frac{\lambda_{\min}}{G} = \frac{\lambda_{\min}}{10}, \ldots, \frac{\lambda_{\min}}{30}$$

Rule of thumb / Daumenregel

FD Method – Stability Condition / FD-Methode – Stabilitätsbedingung

Stability Condition for an FD algorithm of 2nd order in space and time– CFL-Condition /
Stabilitätsbedingung für einen FD-Algorithmus zweiter Ordnung in Raum und Zeit– CFL-Bedingung

$$\Delta t \le \frac{1}{\sqrt{D}} \frac{\Delta x}{c}$$

$D = 1,2,3$: **Spatial dimension of the problem / Räumliche Dimension des Problems**

c : **Maximal Energy Propagation Velocity / Maximale Energieausbreitungsgeschwindigkeit**

CFL: Courant, Friedrichs, Lewy / CFL: Courant, Friedrichs, Lewy /

Courant, R., K. Friedrichs und H. Lewy: *Über die partiellen Differenzengleichungen der mathematischen Physik*. Mathematische Annalen, Vol. 100, S. 32-74, 1928. /
Courant, R., K. Friedrichs, and H. Lewy: *On the partial differential equations of mathematical physics*. *IBM Journal*, pp. 215-324, March 1967.

1-D / 1D: $\Delta t \le \Delta t_{\max} = \frac{\Delta x}{c}$ $\qquad \widehat{\Delta t} \le 1$

2-D / 2D: $\Delta t \le \Delta t_{\max} = \frac{1}{\sqrt{2}} \frac{\Delta x}{c}$ $\qquad \widehat{\Delta t} \le \frac{1}{\sqrt{2}} \approx 0.707$

3-D / 3D: $\Delta t \le \Delta t_{\max} = \frac{1}{\sqrt{3}} \frac{\Delta x}{c}$ $\qquad \widehat{\Delta t} \le \frac{1}{\sqrt{3}} \approx 0.577$

$$\widehat{\Delta t} = \frac{\Delta t}{\Delta t_{\text{ref}}} \quad : \quad \textbf{Courant number / Courant-Zahl}$$

$$\Delta t_{\text{ref}} = \frac{\Delta x}{c}$$

FD Method – Normalization / FD-Methode – Normierung

Δx_{ref} = Reference cell width in m / Referenz-Zellenweite in m
c_{ref} = Reference propagation velocity in m/s / Referenz-Ausbreitungsgeschwindigkeit in m/s
ε_{ref} = Reference permittivity in As/Vm / Referenz-Permittivität in As/Vm
E_{ref} = Reference electric field strength in V/m / Elektrische Referenz-Feldstärke in V/m

$$\Delta t = \Delta t_{\text{ref}} \widehat{\Delta t} \qquad \Delta t_{\text{ref}} = \frac{\Delta x_{\text{ref}}}{c_{\text{ref}}}$$

$$\Delta z = \Delta x_{\text{ref}} \widehat{\Delta z}$$

$$c = c_{\text{ref}} \hat{c}$$

$$\varepsilon = \varepsilon_{\text{ref}} \hat{\varepsilon}$$

$$\mu = \mu_{\text{ref}} \hat{\mu} \qquad \mu_{\text{ref}} = c_{\text{ref}}^2 \varepsilon_{\text{ref}}$$

$$E_x = E_{\text{ref}} \hat{E}_x$$

$$J_{\text{ex}} = J_{\text{e ref}} \hat{J}_{\text{ex}} \qquad J_{\text{e ref}} = \frac{\varepsilon_{\text{ref}}}{\Delta t_{\text{ref}}} E_{\text{ref}}$$

FD Method – Normalization / FD-Methode – Normierung

$$E_x^{(n_z,n_t+1)} = 2E_x^{(n_z,n_t)} - E_x^{(n_z,n_t-1)} + \underbrace{c_0^2 \frac{(\Delta t)^2}{(\Delta z)^2}}\left[E_x^{(n_z+1,n_t)} - 2E_x^{(n_z,n_t)} + E_x^{(n_z-1,n_t)}\right] + \underbrace{c_0^2 \mu_0 \Delta t J_{\text{e ref}}}\left[\hat{J}_{\text{ex}}^{(n_z,n_t)} - \hat{J}_{\text{ex}}^{(n_z,n_t-1)}\right]$$

With / Mit

$$\Delta z = \Delta x_{\text{ref}}$$

$$c_{\text{ref}} = c_0 = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$$

$$\varepsilon_{\text{ref}} = \varepsilon_0$$

$$E_{\text{ref}} = 1\ \text{V/m}$$

$$c_0^2 \frac{(\Delta t)^2}{(\Delta z)^2} = c_{\text{ref}}^2 \frac{(\Delta t)^2 (\Delta t_{\text{ref}})^2}{(\Delta x_{\text{ref}})^2} = c_{\text{ref}}^2 \frac{(\widehat{\Delta t})^2 \left(\frac{\Delta x_{\text{ref}}}{c_{\text{ref}}}\right)^2}{(\Delta x_{\text{ref}})^2} = (\widehat{\Delta t})^2$$

$$c_0^2 \mu_0 \Delta t J_{\text{e ref}} = c_{\text{ref}}^2 \mu_{\text{ref}} \widehat{\Delta t} \Delta t_{\text{ref}} \frac{\varepsilon_{\text{ref}}}{\Delta t_{\text{ref}}} E_{\text{ref}} = \widehat{\Delta t}$$

$$\hat{E}_x^{(n_z,n_t+1)} = 2\hat{E}_x^{(n_z,n_t)} - \hat{E}_x^{(n_z,n_t-1)} + (\widehat{\Delta t})^2\left[\hat{E}_x^{(n_z+1,n_t)} - 2\hat{E}_x^{(n_z,n_t)} + \hat{E}_x^{(n_z-1,n_t)}\right] + \widehat{\Delta t}\left[\hat{J}_{\text{ex}}^{(n_z,n_t)} - \hat{J}_{\text{ex}}^{(n_z,n_t-1)}\right]$$

FD Solution of the 1-D Wave Equation / FD-Lösung der 1D Wellengleichung

1-D wave equation / 1D Wellengleichung

$$\frac{\partial^2}{\partial z^2} E_x(z,t) - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} E_x(z,t) = \mu_0 \frac{\partial}{\partial t} J_{\text{ex}}(z,t) \quad \text{for / für} \quad \begin{cases} 0 \le z \le Z \\ 0 \le t \le T \end{cases}$$

Initial condition / Anfangsbedingung

$$E_x(z,t) = J_{\text{ex}}(z,t) = 0 \qquad t \le 0$$

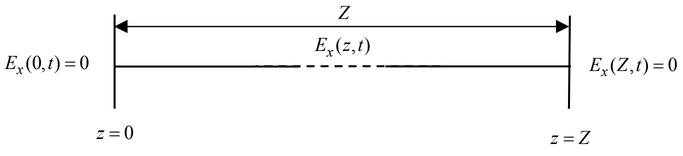
$$J_{\text{ex}}(z,t) = K_{\text{e}0}(z_0)\,\delta(z_0)\,f(t) \qquad t > 0$$

Causality / Kausalität

Boundary condition for a perfectly electrically conducting (PEC) material / Randbedingung für ein ideal elektrisch leitendes Material

$$\left.\begin{aligned} E_x(0,t) &= 0 \\ E_x(Z,t) &= 0 \end{aligned}\right\} \quad \forall t$$

Hyperbolic initial-boundary-value problem / Hyperbolisches Anfangs-Randwert-Problem



FD Solution of the 1-D Wave Equation / FD-Lösung der 1D Wellengleichung

Normalized 1-D FD wave equation / Normierte 1D FD Wellengleichung

$$\hat{E}_x^{(n_z,n_t+1)} = 2\hat{E}_x^{(n_z,n_t)} - \hat{E}_x^{(n_z,n_t-1)} + (\widehat{\Delta t})^2\left[\hat{E}_x^{(n_z+1,n_t)} - 2\hat{E}_x^{(n_z,n_t)} + \hat{E}_x^{(n_z-1,n_t)}\right] \quad \text{for / für} \quad \begin{cases} 1 \le n_z \le N_z \\ 1 \le n_t \le N_t \end{cases}$$

$$+ \widehat{\Delta t}\left[\hat{J}_{\text{ex}}^{(n_z,n_t)} - \hat{J}_{\text{ex}}^{(n_z,n_t-1)}\right]$$

Initial condition / Anfangsbedingung

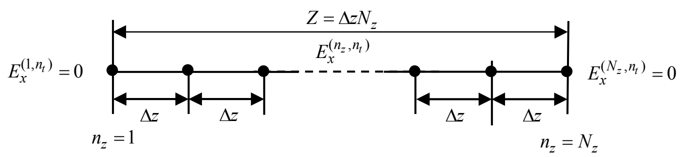
$$E_x^{(n_z,n_t)} = J_{\text{ex}}^{(n_z,n_t)} = 0 \qquad n_t \le 1$$

$$J_{\text{ex}}^{(n_z,n_t)} = K_{\text{ex}}^{(n_{z0})}\,\delta^{(n_{z0})}\,f^{(n_t)} \qquad n_t > 1$$

(Causality / Kausalität)

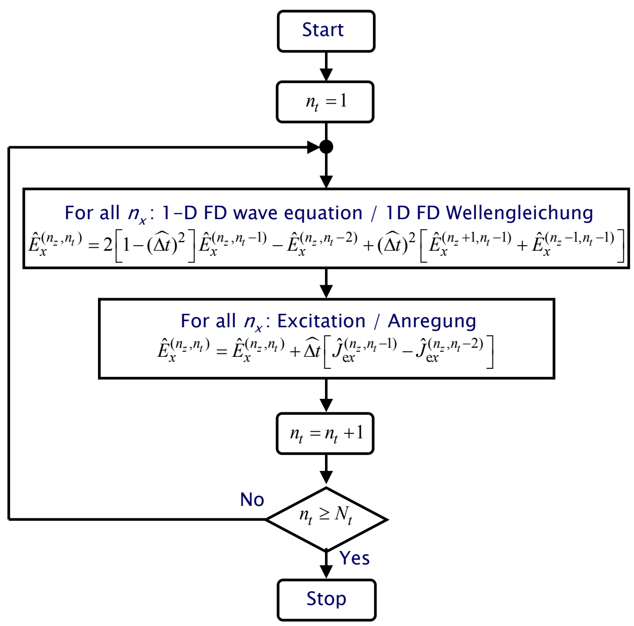
Boundary condition / Randbedingung

$$\left.\begin{aligned} E_x^{(1,n_t)} &= 0 \\ E_x^{(N_z,n_t)} &= 0 \end{aligned}\right\} \quad 1 \le n_t \le N_t$$



Discrete hyperbolic initial-boundary-value problem / Diskretes hyperbolisches Anfangs-Randwert-Problem

FD Method – 1D FD Wave Equation – Flow Chart / FD-Methode – 1D FD-Wellengleichung – Flussdiagramm



FD Method – 1D Wave Equation – Poynting Vector – Energy Density Flow / FD-Methode – 1D Wellengleichung – Poynting-Vektor – Energiedichtefluss

$$\underline{\mathbf{S}}_{\mathrm{em}}(\underline{\mathbf{R}},t)=\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\times\underline{\mathbf{H}}(\underline{\mathbf{R}},t)$$

$$S_{\mathrm{em}\,z}(z,t)=E_x(z,t)\,H_y(z,t)$$

$$\frac{\partial}{\partial t}H_y(z,t)=-\frac{1}{\mu_0}\frac{\partial}{\partial z}E_x(z,t)$$

$$H_y(z,t)=\underbrace{-\frac{1}{\mu_0}\frac{\partial}{\partial z}E_x(z,t_0)}_{=H_y(z,t_0)}-\frac{1}{\mu_0}\int_{t'=t_0}^{t}\frac{\partial}{\partial z}E_x(z,t')dt'$$

$$=H_y(z,t_0)-\frac{1}{\mu_0}\int_{t'=t_0}^{t}\frac{\partial}{\partial z}E_x(z,t')dt'$$

$$\frac{\partial}{\partial t}H_y(z,t)=H_y(z,t_0)-\frac{1}{\mu_0}\int_{t'=t_0}^{t_0+\Delta t}\frac{\partial}{\partial z}E_x(z,t')dt'$$

$$=H_y(z,t_0)-\frac{1}{\mu_0}\frac{\partial}{\partial z}E_x\left(z,t_0+\frac{\Delta t}{2}\right)\underbrace{\int_{t'=t_0}^{t_0+\Delta t}dt'}_{=\Delta t}$$

Applying the mid-point rule / Wende die Mittelpunktsregel an

$$\approx H_y(z,t_0)-\frac{\Delta t}{\mu_0}\underbrace{\frac{\partial}{\partial z}E_x\left(z,t_0+\frac{\Delta t}{2}\right)}_{=\frac{1}{2\Delta z}\left[E_x\left(z+\Delta z,t_0+\frac{\Delta t}{2}\right)-E_x\left(z-\Delta z,t_0\frac{\Delta t}{2}\right)\right]}$$

FD Method – 1D Wave Equation – Poynting Vector – Energy Density Flow / FD-Methode – 1D Wellengleichung – Poynting-Vektor – Energiedichtefluss

$$\frac{\partial}{\partial t}H_y(z,t)\approx H_y(z,t_0)-\frac{1}{2}\frac{1}{\mu_0}\frac{\Delta t}{\Delta z}\left[E_x\left(z+\Delta z,t_0+\frac{\Delta t}{2}\right)-E_x\left(z-\Delta z,t_0\frac{\Delta t}{2}\right)\right]$$

$\Delta z=\Delta x_{\mathrm{ref}}$ $\qquad \Delta t=\Delta t_{\mathrm{ref}}\widehat{\Delta t}$ $\qquad \Delta t_{\mathrm{ref}}=\frac{\Delta z}{c_0}$

$c_{\mathrm{ref}}=c_0=\frac{1}{\sqrt{\varepsilon_0\mu_0}}$ $\qquad c=c_0\hat{c}$

$\varepsilon_{\mathrm{ref}}=\varepsilon_0$ $\qquad \mu=\mu_{\mathrm{ref}}\hat{\mu}$ $\qquad \mu_{\mathrm{ref}}=\mu_0$

$E_{\mathrm{ref}}=1\ \mathrm{V/m}$ $\qquad E_x=E_{\mathrm{ref}}\hat{E}_x$

$$H_y=H_{\mathrm{ref}}\hat{H}_y \qquad H_{\mathrm{ref}}=\frac{E_{\mathrm{ref}}}{c_{\mathrm{ref}}\mu_{\mathrm{ref}}}=\frac{\sqrt{\varepsilon_{\mathrm{ref}}\mu_{\mathrm{ref}}}}{\mu_{\mathrm{ref}}}E_{\mathrm{ref}}=\sqrt{\frac{\varepsilon_{\mathrm{ref}}}{\mu_{\mathrm{ref}}}}E_{\mathrm{ref}}=\frac{E_{\mathrm{ref}}}{Z_{\mathrm{ref}}}$$

$$S_{\mathrm{em\,z}}=S_{\mathrm{em\,ref}}\hat{S}_{\mathrm{em\,z}} \qquad S_{\mathrm{em\,ref}}=E_{\mathrm{ref}}H_{\mathrm{ref}}=\frac{E_{\mathrm{ref}}^2}{Z_{\mathrm{ref}}}$$

$$\sqrt{\frac{\varepsilon_{\mathrm{ref}}}{\mu_{\mathrm{ref}}}}E_{\mathrm{ref}}H_y(z,t)\approx\sqrt{\frac{\varepsilon_{\mathrm{ref}}}{\mu_{\mathrm{ref}}}}E_{\mathrm{ref}}H_y(z,t_0)-\frac{1}{2}\frac{1}{\mu_{\mathrm{ref}}}\frac{\widehat{\Delta t}}{\Delta z}\frac{\Delta z}{c_{\mathrm{ref}}}E_{\mathrm{ref}}\left[\hat{E}_x\left(z+\Delta z,t_0+\frac{\Delta t}{2}\right)-\hat{E}_x\left(z-\Delta z,t_0\frac{\Delta t}{2}\right)\right]$$

$$H_y(z,t)\approx H_y(z,t_0)-\frac{1}{2}\sqrt{\frac{\mu_{\mathrm{ref}}}{\varepsilon_{\mathrm{ref}}}}\frac{1}{E_{\mathrm{ref}}}\frac{1}{\mu_{\mathrm{ref}}}\frac{\widehat{\Delta t}}{\Delta z}\frac{\Delta z}{c_{\mathrm{ref}}}E_{\mathrm{ref}}\left[\hat{E}_x\left(z+\Delta z,t_0+\frac{\Delta t}{2}\right)-\hat{E}_x\left(z-\Delta z,t_0\frac{\Delta t}{2}\right)\right]$$

$$H_y(z,t)\approx H_y(z,t_0)-\frac{\widehat{\Delta t}}{2}\frac{1}{\sqrt{\varepsilon_{\mathrm{ref}}\mu_{\mathrm{ref}}}}\frac{1}{c_{\mathrm{ref}}}\left[\hat{E}_x\left(z+\Delta z,t_0+\frac{\Delta t}{2}\right)-\hat{E}_x\left(z-\Delta z,t_0\frac{\Delta t}{2}\right)\right]$$

FD Method – 1D Wave Equation – Poynting Vector – Energy Density Flow / FD-Methode – 1D Wellengleichung – Poynting-Vektor – Energiedichtefluss

$$\hat{H}_y(z,t) \approx \hat{H}_y(z,t_0) - \frac{\widehat{\Delta t}}{2} \frac{1}{\sqrt{\varepsilon_{\text{ref}} \mu_{\text{ref}}}} \frac{1}{c_{\text{ref}}} \left[\hat{E}_x\left(z+\Delta z, t_0 + \frac{\Delta t}{2}\right) - \hat{E}_x\left(z-\Delta z, t_0 \frac{\Delta t}{2}\right) \right]$$

$$\hat{H}_y(z,t) \approx \hat{H}_y(z,t_0) - \frac{\widehat{\Delta t}}{2} \left[\hat{E}_x\left(z+\Delta z, t_0 + \frac{\Delta t}{2}\right) - \hat{E}_x\left(z-\Delta z, t_0 \frac{\Delta t}{2}\right) \right]$$

$$\hat{H}_y^{(n_z,n_t)} = \hat{H}_y^{(n_z,n_t-1)} - \frac{\widehat{\Delta t}}{2} \left[\hat{E}_x^{(n_z+1,n_t)} - \hat{E}_x^{(n_z-1,n_t)} \right]$$

$$\underline{\mathbf{S}}_{\text{em}}(\underline{\mathbf{R}},t) = \underline{\mathbf{E}}(\underline{\mathbf{R}},t) \times \underline{\mathbf{H}}(\underline{\mathbf{R}},t)$$

$$S_{\text{em}\,z}(z,t) = E_x(z,t)\, H_y(z,t)$$

$$\hat{S}_{\text{em}\,y}^{(n_z,n_t)} = \hat{H}_y^{(n_z,n_t)} \hat{E}_x^{(n_z,n_t)}$$

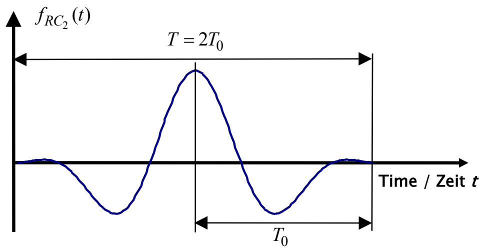
FD Method – 1D Wave Equation – Example / FD-Methode – 1D Wellengleichung – Beispiel

Raised cosine pulse with n cycles /
Aufsteigender Kosinus-Impuls mit n Zyklen

$$f_{RC_n}(t) = \begin{cases} \frac{(-1)^n}{2}\left[1-\cos\left(\frac{2\pi f_0}{n} t\right)\right]\cos(2\pi f_0 t) & 0 < t < \frac{n}{f_0} = nT_0 = T \\ 0 & \text{else / sonst} \end{cases}$$

Raised cosine pulse with 2 cycles /
Aufsteigender Kosinus-Impuls mit 2 Zyklen

$$f_{RC_2}(t) = \begin{cases} \frac{1}{2}\left[1-\cos(\pi f_0 t)\right]\cos(2\pi f_0 t) & 0 < t < \frac{2}{f_0} = 2T_0 = T \\ 0 & \text{else / sonst} \end{cases}$$



Frequency / Frequenz

$$f_0 = \frac{1}{T_0}$$

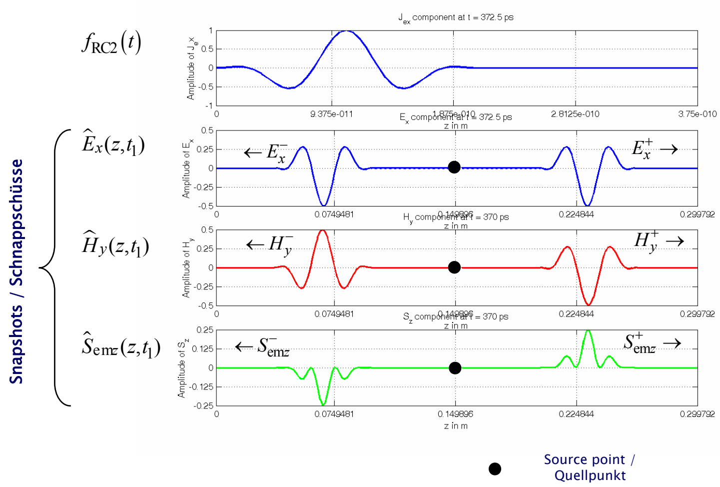
Circular Frequency /
Kreisfrequenz

$$\omega_0 = \frac{2\pi}{T_0}$$

FD Method – 1-D Wave Equation – Example / FD-Methode – 1D Wellengleichung – Beispiel

Electric current density excitation: broadband pulse /
Elektrische Stromdichteanregung: breitbandiger Impuls

$$J_{ex}(z=z_0,t) \sim f_{RC2}(t) \rightarrow E_x(z,t) \sim f_{RC2}\left[t \mp \frac{z-z_0}{c_0}\right]$$



End of Lecture 3 / Ende der 3. Vorlesung