

**Numerical Methods of  
Electromagnetic Field Theory I (NFT I)  
Numerische Methoden der  
Elektromagnetischen Feldtheorie I (NFT I) /**

**4th Lecture / 4. Vorlesung**

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**FD Solution of the 1-D Wave Equation /  
FD-Lösung der 1D Wellengleichung**

**Normalized 1-D FD wave equation / Normierte 1D FD Wellengleichung**

$$\hat{E}_x^{(n_z, n_t+1)} = 2\hat{E}_x^{(n_z, n_t)} - \hat{E}_x^{(n_z, n_t-1)} + (\Delta t)^2 \left[ \hat{E}_x^{(n_z+1, n_t)} - 2\hat{E}_x^{(n_z, n_t)} + \hat{E}_x^{(n_z-1, n_t)} \right] \quad \text{for / für } \begin{cases} 1 \leq n_z \leq N_z \\ 1 \leq n_t \leq N_t \end{cases}$$

$$+ \Delta t \left[ \hat{j}_{\text{ex}}^{(n_z, n_t)} - \hat{j}_{\text{ex}}^{(n_z, n_t-1)} \right]$$

**Initial condition / Anfangsbedingung**

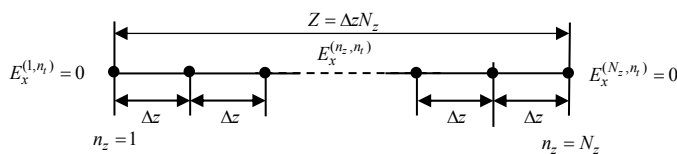
$$E_x^{(n_z, n_t)} = j_{\text{ex}}^{(n_z, n_t)} = 0 \quad n_t \leq 1$$

$$j_{\text{ex}}^{(n_z, n_t)} = K_{\text{ex}}^{(n_{z0})} \delta^{(n_{z0})} f^{(n_t)} \quad n_t > 1$$

**(Causality /  
Kausalität)**

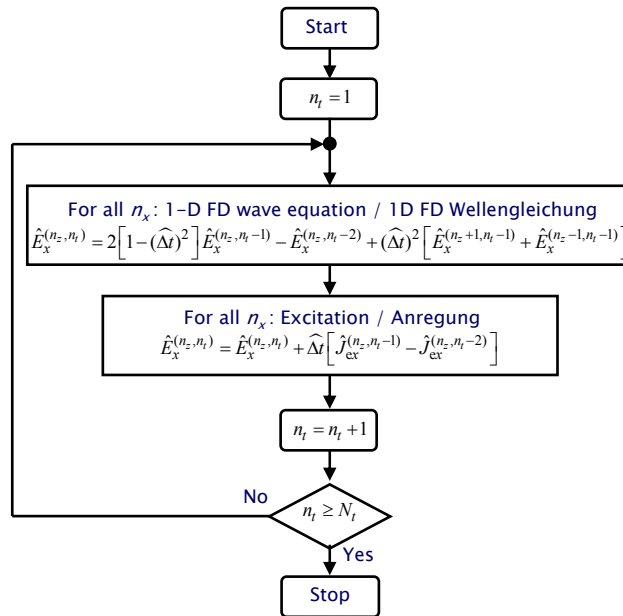
**Boundary condition / Randbedingung**

$$\left. \begin{aligned} E_x^{(1, n_t)} &= 0 \\ E_x^{(N_z, n_t)} &= 0 \end{aligned} \right\} 1 \leq n_t \leq N_t$$



**Discrete hyperbolic  
initial-boundary-value  
problem /  
Diskretes  
hyperbolsches  
Anfangs-Randwert-  
Problem**

**FD Method - 1-D FD Wave Equation - Flow Chart /  
FD-Methode - 1D FD-Wellengleichung - Flussdiagramm**



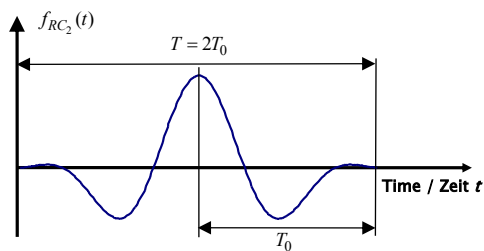
**FD Method - 1-D Wave Equation - Example /  
FD-Methode - 1D Wellengleichung - Beispiel**

**Raised cosine pulse with  $n$  cycles /  
Aufsteigender Kosinus-Impuls mit  $n$  Zyklen**

$$f_{RC_n}(t) = \begin{cases} \frac{(-1)^n}{2} \left[ 1 - \cos\left(\frac{2\pi f_0 t}{n}\right) \right] \cos(2\pi f_0 t) & 0 < t < \frac{n}{f_0} = nT_0 = T \\ 0 & \text{else / sonst} \end{cases}$$

**Raised cosine pulse with 2 cycles /  
Aufsteigender Kosinus-Impuls mit 2 Zyklen**

$$f_{RC_2}(t) = \begin{cases} \frac{1}{2} [1 - \cos(\pi ft)] \cos(2\pi f_0 t) & 0 < t < \frac{2}{f_0} = 2T_0 = T \\ 0 & \text{else / sonst} \end{cases}$$



**Frequency / Frequenz**

$$f_0 = \frac{1}{T_0}$$

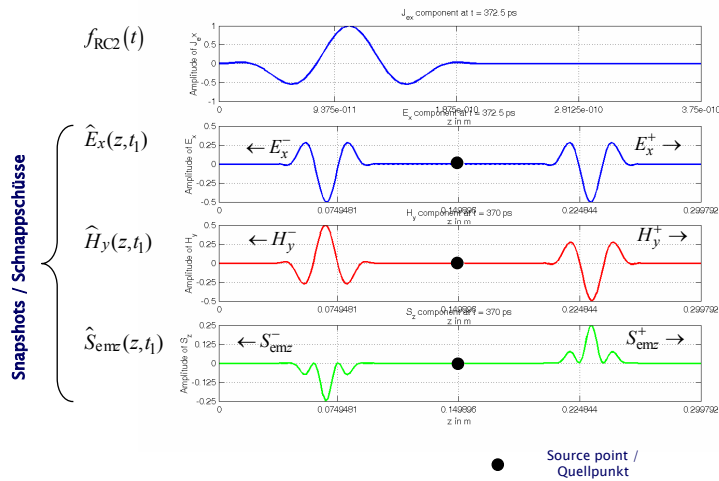
**Circular Frequency /  
Kreisfrequenz**

$$\omega_0 = \frac{2\pi}{T_0}$$

## FD Method - 1-D Wave Equation - Example / FD-Methode - 1D Wellengleichung - Beispiel

Electric current density excitation: broadband pulse /  
Elektrische Stromdichteanregung: breitbandiger Impuls

$$J_{ex}(z=z_0, t) \sim f_{RC2}(t) \rightarrow E_x(z, t) \sim f_{RC2} \left[ t \mp \frac{z-z_0}{c_0} \right]$$



## Numerical Results - Validation / Numerische Ergebnisse - Validierung

Numerical Results / Numerische Ergebnisse



Validation / Validierung

Compare numerical results with analytical solutions or with other  
numerical solutions. / Vergleiche die numerischen Ergebnisse mit  
analytischen Lösungen oder anderen numerischen Lösungen

**Numerical Results - Validation /  
Numerische Ergebnisse - Validierung**

**1. Plane Wave Solution of the Homogeneous Case -  
No sources, no boundaries! /  
Ebene Wellen als Lösung des homogenen Falles -  
Keine Quellen, keine Ränder!**

*Gives the correct characteristic, but not the correct amplitude and  
no reflections at the boundaries! /  
Gibt die korrekte Charakteristik, aber nicht die korrekte Amplitude und keine  
Reflexionen an den Rändern wieder!*

**2. Green's Function Solution of the Inhomogeneous Case -  
"Point" source, but no boundaries,  
if we use the free-space Green's function! /  
Lösung über Greensche Funktion für den inhomogenen Fall -  
„Punkt“quelle, aber keine Ränder, wenn wir die  
Greensche Funktion für den Freiraum verwenden!**

*Gives the correct characteristic and correct amplitude, but no reflections  
at the boundaries! /  
Gibt die korrekte Charakteristik und die korrekte Amplitude, aber keine  
Reflexionen an den Rändern wieder!*

**FD Method - 1-D Wave Equation - Example /  
FD-Methode - 1D Wellengleichung - Beispiel**

Homogeneous scalar 1-D wave equation for  
the electric field strength / Homogene, skalare  
1D-Wellengleichung für die elektrische  
Feldstärke

$$\frac{\partial^2}{\partial z^2} E_x(z,t) - \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} E_x(z,t) = 0$$

Splitting of the 1D wave operator /  
Aufspaltung des 1D-Wellenoperators

$$\left( \frac{\partial^2}{\partial z^2} - \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} \right) E_x(z,t) = 0$$

$$\left( \frac{\partial}{\partial z} + \frac{1}{c_0} \frac{\partial}{\partial t} \right) \left( \frac{\partial}{\partial z} - \frac{1}{c_0} \frac{\partial}{\partial t} \right) E_x(z,t) = 0$$

Hyperbolic partial differential equation /  
Hperbolische partielle Differentialgleichung

$$\left( \frac{\partial}{\partial z} + \frac{1}{c_0} \frac{\partial}{\partial t} \right) \left( \frac{\partial}{\partial z} - \frac{1}{c_0} \frac{\partial}{\partial t} \right) E_x(z,t) = 0$$

One-way wave equation /  
"One-way" Wellengleichung

$$\left( \frac{\partial}{\partial z} + \frac{1}{c_0} \frac{\partial}{\partial t} \right) E_x(z,t) = 0$$

$$\left( \frac{\partial}{\partial z} - \frac{1}{c_0} \frac{\partial}{\partial t} \right) E_x(z,t) = 0$$

## FD Method – 1-D Wave Equation – Example / FD-Methode – 1D Wellengleichung – Beispiel

$$\begin{aligned} \left(\frac{\partial}{\partial z} + \frac{1}{c_0} \frac{\partial}{\partial t}\right) E_x(z,t) &= \left(\frac{\partial}{\partial z} + \frac{1}{c_0} \frac{\partial}{\partial t}\right) E_0^+ \left(z, t - \frac{z}{c_0}\right) \\ &= \frac{\partial}{\partial z} E_0^+ \left(z, t - \frac{z}{c_0}\right) + \frac{1}{c_0} \frac{\partial}{\partial t} E_0^+ \left(z, t - \frac{z}{c_0}\right) \\ &= -\frac{1}{c_0} E_0^+ \left(z, t - \frac{z}{c_0}\right) + \frac{1}{c_0} E_0^+ \left(z, t - \frac{z}{c_0}\right) \\ &= 0 \end{aligned}$$

$$\begin{aligned} \left(\frac{\partial}{\partial z} - \frac{1}{c_0} \frac{\partial}{\partial t}\right) E_x(z,t) &= \left(\frac{\partial}{\partial z} - \frac{1}{c_0} \frac{\partial}{\partial t}\right) E_0^- \left(z, t + \frac{z}{c_0}\right) \\ &= \frac{\partial}{\partial z} E_0^- \left(z, t + \frac{z}{c_0}\right) - \frac{1}{c_0} \frac{\partial}{\partial t} E_0^- \left(z, t + \frac{z}{c_0}\right) \\ &= \frac{1}{c_0} E_0^- \left(z, t + \frac{z}{c_0}\right) - \frac{1}{c_0} E_0^- \left(z, t + \frac{z}{c_0}\right) \\ &= 0 \end{aligned}$$

## FD Method – 1-D Wave Equation – Example / FD-Methode – 1D Wellengleichung – Beispiel

Homogeneous scalar 1-D wave equation for  
the electric field strength / Homogene, skalare  
1D-Wellengleichung für die elektrische  
Feldstärke

$$\frac{\partial^2}{\partial z^2} E_x(z,t) - \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} E_x(z,t) = 0$$

$$\left(\frac{\partial}{\partial z} + \frac{1}{c_0} \frac{\partial}{\partial t}\right) \left(\frac{\partial}{\partial z} - \frac{1}{c_0} \frac{\partial}{\partial t}\right) E_x(z,t) = 0$$

Solution is a left and right propagating plane wave /  
Lösung ist eine nach links und rechts laufende  
ebene Welle

$$\begin{aligned} E_x(z,t) &= E_0 \left(z, t \mp \frac{z}{c_0}\right) \\ &= E_0^+ \left(z, t - \frac{z}{c_0}\right) + E_0^- \left(z, t + \frac{z}{c_0}\right) \end{aligned}$$

A wave, which  
propagates for  
increasing time  $t$  in  
positive  $z$  direction /  
Eine Welle, die sich  
für zunehmende Zeit  
 $t$  in positive  $z$ -  
Richtung ausbreitet

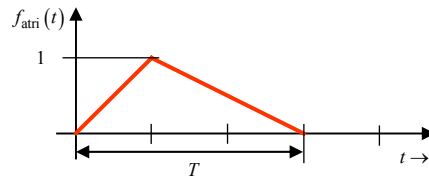
A wave, which  
propagates for  
increasing time  $t$  in  
negative  $z$  direction /  
Eine Welle, die sich  
für zunehmende Zeit  
 $t$  in negative  $z$ -  
Richtung ausbreitet

## FD Method – 1-D Wave Equation – Example / FD-Methode – 1D Wellengleichung – Beispiel

Consider an asymmetric triangular pulse /  
Betrachte einen asymmetrischen  
Dreiecksimpuls

$$E_x(z = z_0, t) = E_0 f_{\text{atri}}(t)$$

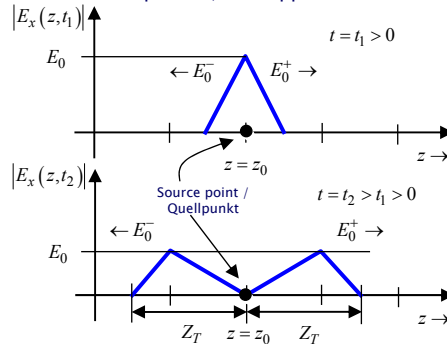
Excitation function / Anregungsfunktion



This means, that the solution for all  $z$  and  $t$  is  
given by / Dies bedeutet, dass die Lösung für  
alle  $z$  und  $t$  gegeben ist durch

$$\begin{aligned} E_x(z, t) &= E_0 \left( z, t \mp \frac{z - z_0}{c_0} \right) \\ &= E_0 f_{\text{atri}} \left( z, t \mp \frac{z - z_0}{c_0} \right) \\ &= E_0 f_{\text{atri}} \left( z, t - \frac{z - z_0}{c_0} \right) + E_0 f_{\text{atri}} \left( z, t + \frac{z - z_0}{c_0} \right) \\ &= E_0^+ \left( z, t - \frac{z - z_0}{c_0} \right) + E_0^- \left( z, t + \frac{z - z_0}{c_0} \right) \end{aligned}$$

Snapshots / Schnappschüsse



## FD Method – 1-D Wave Equation – Example / FD-Methode – 1D Wellengleichung – Beispiel

$$E_x(z, t) = E_0 f_{\text{atri}} \left( z, t \mp \frac{z - z_0}{c_0} \right)$$

$$t \mp \frac{z - z_0}{c_0} = 0$$

$$z_1^+(t_1) = z_0 + c_0 t_1$$

$$z_1^-(t_1) = z_0 - c_0 t_1$$

$$t = \pm \frac{z - z_0}{c_0}$$

$$c_0 t = \pm z \mp z_0$$

$$\pm c_0 t = z - z_0$$

$$z(t) = z_0 \pm c_0 t$$

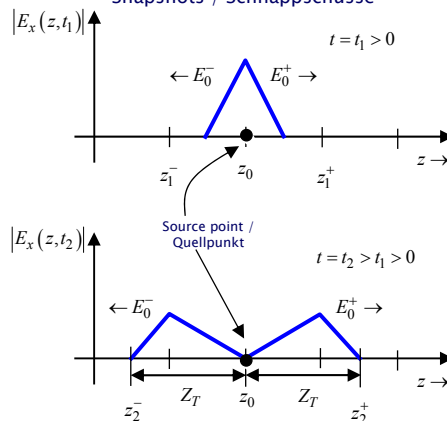
$$z_2^+(t_2) = z_0 + c_0 t_2$$

$$z_2^-(t_2) = z_0 - c_0 t_2$$

$$z^+(t) = z_0 + c_0 t$$

$$z^-(t) = z_0 - c_0 t$$

Snapshots / Schnappschüsse



**FD Method – 1-D Wave Equation – Example /  
FD-Methode – 1D Wellengleichung – Beispiel**

$$\frac{\partial^2}{\partial z^2} E_x(z,t) - \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} E_x(z,t) = 0$$

$$E_x(z,t) = E_0 \left( z, t \mp \frac{z}{c_0} \right)$$

$$\frac{\partial}{\partial t} H_y(z,t) = -\frac{1}{\mu_0} \frac{\partial}{\partial z} E_x(z,t) \quad ?$$

$$S_{\text{emz}}(z,t) = E_x(z,t) H_y(z,t)$$

**FD Method – 1-D Helmholtz Equation (Reduced Wave Equation)  
FD-Methode – 1D Helmholtz-Gleichung (Schwingungsgleichung)**

Homogeneous scalar 1-D wave equation /  
Homogene, skalare 1D-Wellengleichung

$$\frac{\partial^2}{\partial z^2} E_x(z,t) - \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} E_x(z,t) = 0$$

1-D Fourier transform with  
regard to time  $t$  /  
1D Fourier-Transformation  
bezüglich der Zeit  $t$

$$E_x(z, \omega) = \int_{t=-\infty}^{\infty} E_x(z, t) e^{j\omega t} dt$$

$$= FT_t \{ E_x(z, t) \}$$

$$E_x(z, \omega) \bullet \longleftrightarrow E_x(z, t)$$

1-D inverse Fourier transform with  
regard to circular frequency  $\omega$  /  
1D inverse Fourier-Transformation  
bezüglich der Kreisfrequenz  $\omega$

$$E_x(z, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E_x(z, \omega) e^{-j\omega t} d\omega$$

$$= FT_{\omega}^{-1} \{ E_x(z, \omega) \}$$

$$E_x(z, t) \circ \bullet E_x(z, \omega)$$

$$E_x(z, \omega) \bullet \longleftrightarrow E_x(z, t)$$

$$-j\omega \bullet \longleftrightarrow \frac{\partial}{\partial t}$$

$$-\omega^2 \bullet \longleftrightarrow \frac{\partial^2}{\partial t^2}$$

## FD Method – 1-D Helmholtz Equation (Reduced Wave Equation) FD-Methode – 1D Helmholtz-Gleichung (Schwingungsgleichung)

Homogeneous scalar 1-D wave equation /  
Homogene, skalare 1D-Wellengleichung

$$\frac{\partial^2}{\partial z^2} E_x(z, t) - \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} E_x(z, t) = 0$$

Solution in the time domain /  
Lösung im Zeitbereich

$$E_x(z, t) = E_0 \left( z, t \mp \frac{z}{c_0} \right)$$

$$E_x(z, t) \circ \rightarrow E_x(z, \omega)$$

$$\frac{\partial^2}{\partial t^2} \circ \rightarrow -\omega^2$$

Homogeneous scalar 1-D Helmholtz wave  
equation (reduced wave equation) /  
Homogene, skalare 1D Helmholtz-Gleichung  
(Schwingungsgleichung)

$$\frac{\partial^2}{\partial z^2} E_x(z, \omega) - \frac{1}{c_0^2} (-\omega^2) E_x(z, \omega) = 0$$

$$\frac{\partial^2}{\partial z^2} E_x(z, \omega) + \underbrace{\frac{\omega^2}{c_0^2}}_{=k_0^2} E_x(z, \omega) = 0$$

$$\frac{\partial^2}{\partial z^2} E_x(z, \omega) + k_0^2 E_x(z, \omega) = 0$$

Solution in the frequency domain /  
Lösung im Frequenzbereich

$$E_x(z, \omega) = E_0(\omega) e^{\pm jk_0 z}$$

## FD Method – 1-D Helmholtz Equation (Reduced Wave Equation) FD-Methode – 1D Helmholtz-Gleichung (Schwingungsgleichung)

Maxwell's equations in the time  
domain / Maxwell'sche  
Gleichungen im Zeitbereich

$$\frac{\partial}{\partial t} H_y(z, t) = -\frac{1}{\mu_0} \frac{\partial}{\partial z} E_x(z, t)$$

$$\frac{\partial}{\partial t} E_x(z, t) = -\frac{1}{\epsilon_0} \frac{\partial}{\partial z} H_y(z, t)$$

Maxwell's equations in the  
frequency domain /  
Maxwell'sche Gleichungen im  
Frequenzbereich

$$-j\omega H_y(z, \omega) = -\frac{1}{\mu_0} \frac{\partial}{\partial z} E_x(z, \omega)$$

$$-j\omega E_x(z, \omega) = -\frac{1}{\epsilon_0} \frac{\partial}{\partial z} H_y(z, \omega)$$

$$E_x(z, t) \circ \rightarrow E_x(z, \omega)$$

$$H_y(z, t) \circ \rightarrow H_y(z, \omega)$$

$$\frac{\partial}{\partial t} \circ \rightarrow j\omega$$

Electric field strength: plane wave /  
Elektrische Feldstärke: ebene Welle

$$E_x(z, \omega) = E_0(\omega) e^{\pm jk_0 z}$$

$$H_y(z, \omega) = \frac{1}{j\omega\mu_0} \frac{\partial}{\partial z} E_x(z, \omega) = \frac{1}{j\omega\mu_0} \frac{\partial}{\partial z} E_0(\omega) e^{\pm jk_0 z} = \frac{1}{j\omega\mu_0} E_0(\omega) \frac{\partial}{\partial z} e^{\pm jk_0 z} = \pm \frac{jk_0}{j\omega\mu_0} E_0(\omega) e^{\pm jk_0 z}$$

$$= \pm \frac{\underbrace{\omega/c_0 = \omega\sqrt{\epsilon_0\mu_0}}_{\tilde{k}_0}}{\omega\mu_0} E_0(\omega) e^{\pm jk_0 z} = \pm \frac{1}{Z_0} E_0(\omega) e^{\pm jk_0 z}$$

**Magnetic field strength: plane wave /  
Magnetische Feldstärke: ebene Welle**



## FD Method – 1-D Helmholtz Equation (Reduced Wave Equation) FD-Methode – 1D Helmholtz-Gleichung (Schwingungsgleichung)

Homogeneous scalar 1-D wave equation in the time domain / Homogene, skalare 1D-Wellengleichung im Zeitbereich

$$\frac{\partial^2}{\partial z^2} E_x(z,t) - \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} E_x(z,t) = 0$$

$$\frac{\partial^2}{\partial z^2} H_y(z,t) - \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} H_y(z,t) = 0$$

Solution of the 1-D wave equation in the time domain / Lösung der homogenen 1D-Wellengleichung im Zeitbereich

$$E_x(z,t) = E_0 \left( z, t \mp \frac{z}{c_0} \right)$$

$$H_y(z,t) = H_0 \left( z, t \mp \frac{z}{c_0} \right)$$

Homogeneous, scalar 1-D Helmholtz equation in the frequency domain / Homogene, skalare 1D-Helmholtz-Gleichung im Frequenzbereich

$$\frac{\partial^2}{\partial z^2} E_x(z,\omega) + k_0^2 E_x(z,\omega) = 0$$

$$\frac{\partial^2}{\partial z^2} H_y(z,\omega) + k_0^2 H_y(z,\omega) = 0$$

Solution of the 1-D Helmholtz equation in the frequency domain / Lösung der homogenen 1D-Helmholtz-Gleichung im Frequenzbereich

$$E_x(z,\omega) = E_0(\omega) e^{\pm jk_0 z}$$

$$H_y(z,\omega) = H_0(\omega) e^{\pm jk_0 z}$$

$$= \pm \frac{1}{Z_0} E_0(\omega) e^{\pm jk_0 z}$$

Solution of the 1-D wave equation for the magnetic field strength in terms of the electric field strength / Lösung der homogenen 1D-Wellengleichung für die magnetische Feldstärke als Funktion der elektrischen Feldstärke

$$H_y(z,t) = \pm \frac{1}{Z_0} E_0 \left( z, t \mp \frac{z}{c_0} \right)$$

## FD Method – 1-D Wave Equation – Example / FD-Methode – 1D Wellengleichung – Beispiel

Homogeneous scalar 1-D wave equations / Homogene, skalare 1D-Wellengleichungen

$$\frac{\partial^2}{\partial z^2} E_x(z,t) - \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} E_x(z,t) = 0$$

$$\frac{\partial^2}{\partial z^2} H_y(z,t) - \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} H_y(z,t) = 0$$

Solutions / Lösungen

$$E_x(z,t) = E_0 \left( z, t \mp \frac{z}{c_0} \right)$$

$$H_y(z,t) = \pm \frac{1}{Z_0} E_0 \left( z, t \mp \frac{z}{c_0} \right)$$

Poynting vector / Poynting-Vektor

$$S_{\text{emz}}(z,t) = E_x(z,t) H_y(z,t)$$

$$= E_0 \left( z, t \mp \frac{z}{c_0} \right) H_0 \left( z, t \mp \frac{z}{c_0} \right)$$

$$= E_0 \left( z, t \mp \frac{z}{c_0} \right) \left[ \pm \frac{1}{Z_0} E_0 \left( z, t \mp \frac{z}{c_0} \right) \right]$$

$$= \pm \frac{1}{Z_0} E_0^2 \left( z, t \mp \frac{z}{c_0} \right)$$

$$S_{\text{emz}}(z,t) = \pm \frac{1}{Z_0} E_0^2 \left( z, t \mp \frac{z}{c_0} \right)$$

$$= \frac{1}{Z_0} E_0^2 \left( z, t - \frac{z}{c_0} \right) - \frac{1}{Z_0} E_0^2 \left( z, t + \frac{z}{c_0} \right)$$

$$= \underbrace{\frac{1}{Z_0} E_0^2 \left( z, t - \frac{z}{c_0} \right)}_{S_{\text{emz}}^+(z,t)} - \underbrace{\frac{1}{Z_0} E_0^2 \left( z, t + \frac{z}{c_0} \right)}_{S_{\text{emz}}^-(z,t)}$$

$$= S_{\text{emz}}^+(z,t) + S_{\text{emz}}^-(z,t)$$

Poynting vector of the two plane waves / Poynting-Vektor der beiden ebenen Wellen

$$S_{\text{emz}}^+(z,t) = \frac{1}{Z_0} E_0^2 \left( z, t - \frac{z}{c_0} \right) \quad S_{\text{emz}}^-(z,t) = -\frac{1}{Z_0} E_0^2 \left( z, t + \frac{z}{c_0} \right)$$

## FD Method – 1-D Wave Equation – Example / FD-Methode – 1D Wellengleichung – Beispiel

$$E_x(z, t) = E_0^+ \left( z, t - \frac{z - z_0}{c_0} \right) + E_0^- \left( z, t + \frac{z - z_0}{c_0} \right)$$

$$f_{RC2}(t)$$

$$H_y(z, t) = \left[ \frac{E_0^+ \left( z, t - \frac{z - z_0}{c_0} \right)}{Z_0} \right] + \left[ -\frac{E_0^- \left( z, t + \frac{z - z_0}{c_0} \right)}{Z_0} \right]$$

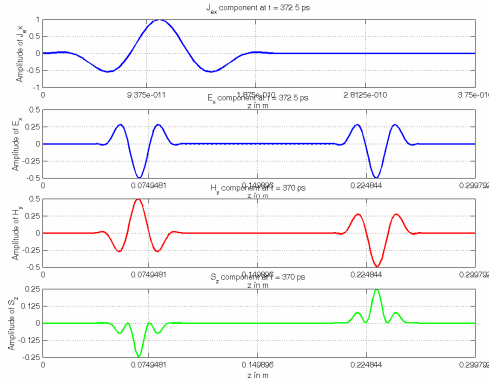
$$\hat{E}_x(z, t)$$

$$\hat{H}_y(z, t)$$

$$S_{\text{emc}}(z, t) = S_{\text{emc}}^+(z, t) + S_{\text{emc}}^-(z, t)$$

$$= \left[ \frac{E_0^+ \left( z, t - \frac{z - z_0}{c_0} \right)}{Z_0} \right] + \left[ -\frac{E_0^- \left( z, t + \frac{z - z_0}{c_0} \right)}{Z_0} \right]$$

$$\hat{S}_{\text{emc}}(z, t)$$



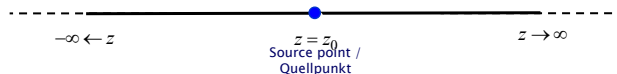
The plane wave solution gives the correct characteristic of the wave field, but the amplitude is not correct! This means we can not verify the numerical results with the plane wave solution of the homogeneous wave equation, because the simulated problem correspond to the solution of the inhomogeneous wave equation. /

Die Ebene-Wellen-Lösung gibt die korrekte Charakteristik des Wellenfeldes wieder, aber die Amplitude der Wellenanteile ist nicht korrekt! Dies bedeutet, dass man die numerischen Resultate mit der Ebenen-Wellen-Lösung nicht vollständig verifizieren kann, da die simulierte Situation mit der Lösung der inhomogenen Wellengleichung korrespondiert.

## Electromagnetic Field of a "Point Source" Excitation in 1-D / Elektromagnetisches Feld einer „Punktquellen“anregung in 1D

We consider a homogeneous infinite 1-D region / Wir betrachten ein homogenes, unendliches 1D-Gebiet

Unknown/Unbekannt:  $E_x(z, \omega) = ?$       $J_{ex}(z = z_0, \omega)$ : Given / Gegeben



where we prescribe an electric current density  $J_{ex}(z, \omega)$  with the unit  $A/m^2$  at  $z = z_0$ , /  
wobei wir eine elektrische Stromdichte mit der Einheit  $A/m^2$  an der Stelle  $z = z_0$  vorgeben.

Then, the unknown electric field strength is a solution of the inhomogeneous Helmholtz equation /  
Die unbekannte elektrische Feldstärke ist dann Lösung der inhomogenen Helmholtz-Gleichung

$$\frac{\partial^2}{\partial z^2} E_x(z, \omega) + k_0^2 E_x(z, \omega) = -j\omega\mu J_{ex}(z, \omega)$$

A solution for the electric field strength is given by the domain Integral representation /  
Eine Lösung für die elektrische Feldstärke ist dann gegeben über die (Gebiets-) Integraldarstellung

$$E_x(z, \omega) = j\omega\mu_0 \int_{z'=-\infty}^{\infty} G(z-z', \omega) J_{ex}(z', \omega) dz'$$

1-D scalar Green's function /  
1D skalare Greensche Funktion

$G(z-z', \omega)$

Convolution integral /  
Faltungintegral

## Electromagnetic Field of a Point Source Excitation in 1-D / Elektromagnetisches Feld einer Punktquellenanregung in 1D

Integral representation /  
Integraldarstellung

$$E_x(z, \omega) = j\omega\mu_0 \int_{z'=-\infty}^{\infty} G(z-z', \omega) J_{ex}(z', \omega) dz'$$

1-D scalar Green's function in the frequency domain /  
1D skalare Greensche Funktion im Frequenzbereich

$$\begin{aligned} G(z-z', \omega) &= \frac{1}{2} \left[ \text{PV} \frac{j}{k_0} + \pi c_0 \delta(z-z') \right] e^{jk_0|z-z'|} \\ &= \frac{1}{2} \left[ \text{PV} \frac{j}{k_0} e^{jk_0|z-z'|} + \pi c_0 \delta(z-z') \right] \\ &= \frac{1}{2} \left[ -\text{PV} \frac{c_0}{j\omega} e^{jk_0|z-z'|} + \pi c_0 \delta(z-z') \right] \\ &= \frac{c_0}{2} \left[ -\text{PV} \frac{1}{j\omega} e^{jk_0|z-z'|} + \pi \delta(z-z') \right] \end{aligned}$$

1-D scalar Green's function  
in the time domain /  
1D skalare Greensche Funktion  
im Zeitbereich

$$G(z, t) = \frac{c_0}{2} u\left(t - \frac{|z|}{c_0}\right)$$

Unit step function /  
Einheitssprungfunktion

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$

Electric current density /  
Elektrische Stromdichte

$$\begin{aligned} J_{ex}(z, \omega) &= \delta(z-z_0) K_{ex}(z, \omega) \\ &= \delta(z-z_0) K_{ex}(z_0, \omega) \end{aligned}$$

Electric surface current density /  
Elektrische Flächenstromdichte

$$K_{ex}(z_0, \omega)$$

Property of the delta-distribution /  
Eigenschaft der Delta-Distribution

$$\delta(z-z_0) f(z) = \delta(z-z_0) f(z_0)$$

## EM Field of a Point Source Excitation in 1-D / EM-Feld einer Punktquellenanregung in 1D

$$\begin{aligned} E_x(z, \omega) &= j\omega\mu_0 \int_{z'=-\infty}^{\infty} G(z-z', \omega) J_{ex}(z', \omega) dz' \\ &= j\omega\mu_0 \int_{z'=-\infty}^{\infty} G(z-z', \omega) \delta(z'-z_0) K_{ex}(z', \omega) dz' \\ &= j\omega\mu_0 G(z-z_0, \omega) K_{ex}(z_0, \omega) \end{aligned}$$

$$\begin{aligned} E_x(z, \omega) &\bullet \rightarrow E_x(z, t) \\ j\omega &\bullet \rightarrow -\frac{\partial}{\partial t} \\ G(z-z_0, \omega) &\bullet \rightarrow G(z-z_0, t) \\ K_{ex}(z_0, \omega) &\bullet \rightarrow K_{ex}(z_0, t) \\ G(z-z_0, \omega) K_{ex}(z_0, \omega) &\bullet \rightarrow G(z-z_0, t) *_t K_{ex}(z_0, t) \end{aligned}$$

The asterisk  $**_t$  denotes convolution  
in time / Der Stern  $**_t$  bezeichnet  
eine Faltung in der Zeit

$$\begin{aligned} E_x(z, t) &= -\mu_0 \frac{\partial}{\partial t} \int_{t'=-\infty}^{\infty} G(z-z_0, t-t') K_{ex}(z_0, t') dt' \\ &= -\frac{c_0\mu_0}{2} \frac{\partial}{\partial t} \int_{t'=-\infty}^{\infty} u\left(t-t' - \frac{|z-z_0|}{c_0}\right) K_{ex}(z_0, t') dt' \\ &= -\frac{c_0\mu_0}{2} \int_{t'=-\infty}^{\infty} \left[ \frac{\partial}{\partial t} u\left(t-t' - \frac{|z-z_0|}{c_0}\right) \right] K_{ex}(z_0, t') dt' \end{aligned}$$

## EM Field of a Point Source Excitation in 1-D / EM-Feld einer Punktquellenanregung in 1D

$$E_x(z,t) = -\frac{c_0 \mu_0}{2} \int_{t'=-\infty}^{\infty} \left[ \frac{\partial}{\partial t} u \left( t-t' - \frac{|z-z_0|}{c_0} \right) \right] K_{\text{ex}}(z_0, t') dt'$$

$$\frac{\partial}{\partial t} u \left( t-t' - \frac{|z-z_0|}{c_0} \right) = \delta \left( t-t' - \frac{|z-z_0|}{c_0} \right)$$

$$\begin{aligned} E_x(z,t) &= -\frac{c_0 \mu_0}{2} \int_{t'=-\infty}^{\infty} \delta \left( t-t' - \frac{|z-z_0|}{c_0} \right) K_{\text{ex}}(z_0, t') dt' \\ &= -\frac{c_0 \mu_0}{2} \int_{t'=-\infty}^{\infty} \delta \left( t' - \left( t - \frac{|z-z_0|}{c_0} \right) \right) K_{\text{ex}}(z_0, t') dt' \\ &= -\frac{c_0 \mu_0}{2} K_{\text{ex}} \left( z_0, t - \frac{|z-z_0|}{c_0} \right) \end{aligned}$$

$$c_0 \mu_0 = Z_0 \approx 377 \, \Omega \quad \text{Wave Impedance of free space (vacuum) / Wellenwiderstand des Freiraumes (Vakuum)}$$

Solution for the x component of the electric field strength /  
Lösung für die x-Komponente der elektrischen Feldstärke

$$E_x(z,t) = -\frac{Z_0}{2} K_{\text{ex}} \left( z_0, t - \frac{|z-z_0|}{c_0} \right)$$

## EM Field of a Point Source Excitation in 1-D / EM-Feld einer Punktquellenanregung in 1D

$$\begin{aligned} H_y(z,\omega) &= \frac{1}{j\omega\mu_0} \frac{\partial}{\partial z} E_x(z,\omega) \\ &= \frac{1}{j\omega\mu_0} \frac{\partial}{\partial z} [j\omega\mu_0 G(z-z_0, \omega) K_{\text{ex}}(z_0, \omega)] \\ &= \frac{\partial}{\partial z} G(z-z_0, \omega) K_{\text{ex}}(z_0, \omega) \end{aligned}$$

$$E_x(z,\omega) = j\omega\mu_0 G(z-z_0, \omega) K_{\text{ex}}(z_0, \omega)$$

$$\frac{\partial}{\partial t} u(t) = \delta(t)$$

$$\frac{\partial}{\partial t} u(-t) = -\delta(t)$$

$$\begin{aligned} H_y(z,t) &= \frac{\partial}{\partial z} \int_{t'=-\infty}^{\infty} G(z-z_0, t-t') K_{\text{ex}}(z_0, t') dt' \\ &= \frac{c_0}{2} \frac{\partial}{\partial z} \int_{t'=-\infty}^{\infty} u \left( t-t' - \frac{|z-z_0|}{c_0} \right) K_{\text{ex}}(z_0, t') dt' \\ &= \frac{c_0}{2} \int_{t'=-\infty}^{\infty} \left[ \frac{\partial}{\partial z} u \left( t-t' - \frac{|z-z_0|}{c_0} \right) \right] K_{\text{ex}}(z_0, t') dt' \\ &= -\frac{c_0}{2} \frac{\text{sgn}(z-z_0)}{c_0} \int_{t'=-\infty}^{\infty} \delta \left( t-t' - \frac{|z-z_0|}{c_0} \right) K_{\text{ex}}(z_0, t') dt' \\ &= -\text{sgn}(z-z_0) \frac{1}{2} K_{\text{ex}} \left( z_0, t - \frac{|z-z_0|}{c_0} \right) \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial z} u \left( t-t' - \frac{|z-z_0|}{c_0} \right) &= \frac{\partial}{\partial z} u \left( t-t' - \frac{\text{sgn}(z-z_0)(z-z_0)}{c_0} \right) \\ &= -\frac{\text{sgn}(z-z_0)}{c_0} \delta \left( t-t' - \frac{\text{sgn}(z-z_0)(z-z_0)}{c_0} \right) \\ &= -\frac{\text{sgn}(z-z_0)}{c_0} \delta \left( t-t' - \frac{|z-z_0|}{c_0} \right) \end{aligned}$$

## EM Field of a Point Source Excitation in 1-D / EM-Feld einer Punktquellenanregung in 1D

Solution for the  $y$  component of the magnetic field strength  
/ Lösung für die  $y$ -Komponente der magnetische Feldstärke

$$H_y(z, t) = -\frac{\text{sgn}(z - z_0)}{2} K_{\text{ex}} \left( z_0, t - \frac{|z - z_0|}{c_0} \right)$$

Solution for the  $x$  component of the electric field strength /  
Lösung für die  $x$ -Komponente der elektrischen Feldstärke

$$E_x(z, t) = -\frac{Z_0}{2} K_{\text{ex}} \left( z_0, t - \frac{|z - z_0|}{c_0} \right)$$

Solution for the  $z$  component of the Poynting vector /  
Lösung für die  $z$ -Komponente des Poynting-Vektors

$$\begin{aligned} S_{\text{em}z}(z, t) &= E_x(z, t)H_y(z, t) \\ &= \left[ -\frac{Z_0}{2} K_{\text{ex}} \left( z_0, t - \frac{|z - z_0|}{c_0} \right) \right] \left[ -\frac{\text{sgn}(z - z_0)}{2} K_{\text{ex}} \left( z_0, t - \frac{|z - z_0|}{c_0} \right) \right] \\ &= \text{sgn}(z - z_0) \frac{Z_0}{4} K_{\text{ex}}^2 \left( z_0, t - \frac{|z - z_0|}{c_0} \right) \end{aligned}$$

## EM Field of a Point Source Excitation in 1-D / EM-Feld einer Punktquellenanregung in 1D

Normalization of the field components /  
Normierung der Feldkomponenten

$$\begin{aligned} \Delta t &= \Delta x_{\text{ref}} \hat{\Delta t} & \Delta x_{\text{ref}} &= \frac{\Delta x_{\text{ref}}}{c_{\text{ref}}} & \Delta z &= \Delta x_{\text{ref}} \hat{\Delta z} & c &= c_{\text{ref}} \hat{c} & \epsilon &= \epsilon_{\text{ref}} \hat{\epsilon} & \mu &= \mu_{\text{ref}} \hat{\mu} & \mu_{\text{ref}} &= \mu_0 \\ E_x &= E_{\text{ref}} \hat{E}_x & H_y &= H_{\text{ref}} \hat{H}_y & H_{\text{ref}} &= \frac{E_{\text{ref}}}{c_{\text{ref}} \mu_{\text{ref}}} = \frac{\sqrt{\epsilon_{\text{ref}} \mu_{\text{ref}}}}{\mu_{\text{ref}}} E_{\text{ref}} = \sqrt{\frac{\epsilon_{\text{ref}}}{\mu_{\text{ref}}}} E_{\text{ref}} = \frac{E_{\text{ref}}}{Z_{\text{ref}}} \\ S_{\text{em}z} &= S_{\text{em ref}} \hat{S}_{\text{em}z} & S_{\text{em ref}} &= E_{\text{ref}} H_{\text{ref}} = \frac{E_{\text{ref}}^2}{Z_{\text{ref}}} & c_{\text{ref}} &= c_0 & \Delta x_{\text{ref}} &= \Delta z & \epsilon_{\text{ref}} &= \epsilon_0 & \mu_{\text{ref}} &= \mu_0 & Z_{\text{ref}} &= Z_0 & K_{\text{e ref}} &= 1 \text{ A/m} \\ J_{\text{ex}} &= J_{\text{e ref}} \hat{J}_{\text{ex}} & J_{\text{e ref}} &= \frac{\epsilon_{\text{ref}}}{\Delta t_{\text{ref}}} E_{\text{ref}} \\ \delta(z) &= \frac{1}{\Delta x_{\text{ref}}} \hat{\delta}(z) \\ K_{\text{ex}} &= K_{\text{e ref}} \hat{K}_{\text{ex}} & K_{\text{e ref}} &= \Delta x_{\text{ref}} J_{\text{e ref}} \end{aligned}$$

EM Field components /  
EM-Feldkomponenten

$$\begin{aligned} K_{\text{ex}}(z_0, t) &= K_{\text{e ref}} f_{RC2}(t) \\ E_x^\pm(z, t) &= -\frac{Z_0}{2} K_{\text{ex}} \left( z_0, t - \frac{|z - z_0|}{c_0} \right) \\ H_y^\pm(z, t) &= \mp \frac{1}{2} K_{\text{ex}} \left( z_0, t - \frac{|z - z_0|}{c_0} \right) \\ S_{\text{em}z}^\pm(z, t) &= \pm \frac{Z_0}{4} \left[ K_{\text{ex}} \left( z_0, t - \frac{|z - z_0|}{c_0} \right) \right]^2 \end{aligned}$$

Normalized EM field components /  
Normierte EM-Feldkomponenten

$$\begin{aligned} \hat{K}_{\text{ex}}(z_0, t) &= f_{RC2}(t) \\ \hat{E}_x^\pm(z, t) &= -\frac{1}{2} \hat{K}_{\text{ex}} \left( z_0, t - \frac{|z - z_0|}{c_0} \right) \\ \hat{H}_y^\pm(z, t) &= \mp \frac{1}{2} \hat{K}_{\text{ex}} \left( z_0, t - \frac{|z - z_0|}{c_0} \right) \\ \hat{S}_{\text{em}z}^\pm(z, t) &= \pm \frac{1}{4} \left[ \hat{K}_{\text{ex}} \left( z_0, t - \frac{|z - z_0|}{c_0} \right) \right]^2 \end{aligned}$$

## FD Method – 1-D Wave Equation – Example / FD-Methode – 1D Wellengleichung – Beispiel

$$\widehat{K}_{\text{ex}}(z_0, t) = f_{\text{RC2}}(t)$$

$$\widehat{E}_x^\pm(z, t) = -\frac{1}{2} \widehat{K}_{\text{ex}} \left( z_0, t - \frac{|z - z_0|}{c_0} \right)$$

$$\widehat{H}_y^\pm(z, t) = \mp \frac{1}{2} \widehat{K}_{\text{ex}} \left( z_0, t - \frac{|z - z_0|}{c_0} \right)$$

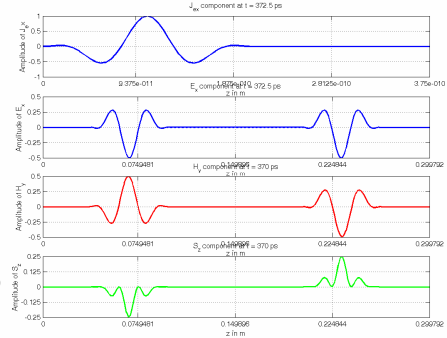
$$\widehat{S}_{\text{emz}}^\pm(z, t) = \pm \frac{1}{4} \left[ \widehat{K}_{\text{ex}} \left( z_0, t - \frac{|z - z_0|}{c_0} \right) \right]^2$$

$$f_{\text{RC2}}(t)$$

$$\widehat{E}_x(z, t)$$

$$\widehat{H}_y(z, t)$$

$$\widehat{S}_{\text{emz}}(z, t)$$



The Green's function method gives the solution of the 1-D simulation area excited by a "point" source, which is in 1-D a singular electric surface current source. The singular source is independent of  $x$  and  $y$ . The reference solution gives the correct characteristic and correct amplitudes. But the solution doesn't account for the reflections at the boundaries, because we used the free-space Green's function. /

Die Methode der Greenschen Funktion ermöglicht die Lösung des vorliegenden Problems, der Anregung des 1D-Simulationsgebietes durch eine „Punkt“quelle, die genauer gesagt in 1D eine singuläre elektrische Flächenstromdichte ist.

Da die singuläre Quelle von  $x$  und  $y$  unabhängig ist. Die Charakteristik und Amplitude stimmt überein, nur die Reflexionen an den Rändern fehlen, was an der Verwendung der Greenschen Funktion für den Freiraum liegt.

## FD Method – Properties / FD-Methode – Eigenschaften

✚ Spatial and Temporal Discretization /  
Räumliche und zeitliche Diskretisierung  $\Delta z = ?$   
 $\Delta t = ?$

✚ Consistency /  
Konsistenz

✚ Dissipation /  
Dissipation

✚ Stability Condition /  
Stabilitätsbedingung  $\Delta t = f(\Delta z)$

✚ Convergence /  
Konvergenz

**Derivation of the Numerical Dispersion Relation for the 1-D FD Scheme of 2nd Order /  
Ableitung der numerischen Dispersionsrelation für das 1D-FD-Schema 2ter Ordnung**

**Stability by the von Neumann's method  
(Fourier series method):**

Insert a complex monofrequent (monochromatic) plane wave into the discrete FD equations and analyze the spectral radius of the amplification matrix, where the spectral radius must be smaller equal one.

**Stabilität durch die von Neumannsche Methode  
(Fourier-Reihen-Methode):**

Setze eine komplex monofrequente (monochromatische) ebene Welle in die diskreten FD-Gleichungen ein und analysiere den spektralen Radius der Verstärkungsmatrix, wobei der spektrale Radius kleinergleich Eins sein muss.

Complex monofrequent (monochromatic) plane wave /  
Komplex monofrequente (monochromatische) ebene Welle

$$E_x(\mathbf{R}, t) = E_0(\omega_0, \hat{\mathbf{k}}) e^{-j(\omega_0 t - \hat{\mathbf{k}} \cdot \mathbf{R})}$$

$$= E_0(\omega_0, \hat{\mathbf{k}}) e^{-j\omega_0 t} e^{j\hat{\mathbf{k}} \cdot \mathbf{R}}$$

$$\{\mathbf{W}\}^{(n+1)} = [\mathbf{G}]_{1D}^{FD} \{\mathbf{W}\}^{(n)} \quad [\mathbf{G}]_{1D}^{FD} : \text{Amplification matrix / Verstärkungsmatrix}$$

Spectral radius /  
Spektraler Radius  $\rho([\mathbf{G}]_{1D}^{FD}) \leq 1$  of the matrix /  
der Matrix  $[\mathbf{G}]_{1D}^{FD}$

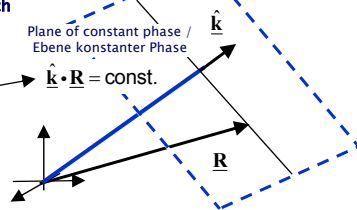
where /  
wobei  $\rho([\mathbf{G}]_{1D}^{FD}) = \max_{n=1, \dots, N} |v_n([\mathbf{G}]_{1D}^{FD})|$   $v_n([\mathbf{G}]_{1D}^{FD})$ :  $n$ th eigenvalue of the matrix /  
 $n$ -ter Eigenwert der Matrix  $[\mathbf{G}]_{1D}^{FD}$

**Derivation of the Stability Condition for the 1-D FD Scheme of 2nd Order /  
Ableitung der Stabilitätsbedingung für das 1D-FD-Schema 2ter Ordnung**

Monofrequent (monochromatic) plane wave in the time domain /  
Monofrequente (monochromatische) ebene Welle im Zeitbereich

$$E_x(\mathbf{R}, t) = E_0(\omega_0, \hat{\mathbf{k}}) e^{-j(\omega_0 t - \hat{\mathbf{k}} \cdot \mathbf{R})}$$

$$= E_0(\omega_0, \hat{\mathbf{k}}) e^{-j\omega_0 t} e^{j\hat{\mathbf{k}} \cdot \mathbf{R}}$$



- Wave vector /  
Wellenvektor  $\mathbf{k} = k_x \mathbf{e}_x + k_y \mathbf{e}_y + k_z \mathbf{e}_z = k_z \mathbf{e}_z$
- Magnitude of the wave vector /  
Betrag des Wellenvektors  $|\mathbf{k}| = \sqrt{\mathbf{k} \cdot \mathbf{k}} = \sqrt{k_x^2 + k_y^2 + k_z^2} = \sqrt{k_z^2} = |k_z| = k$
- Wavenumber /  
Wellenzahl  $k = \frac{\omega_0}{c}$
- Circular frequency /  
Kreisfrequenz  $\omega_0 = 2\pi f_0$
- Propagation direction /  
Ausbreitungsrichtung  $\hat{\mathbf{k}} = \frac{\mathbf{k}}{|\mathbf{k}|} = \frac{k_z \mathbf{e}_z}{|k_z|} = \frac{\text{sgn}(k_z) |k_z| \mathbf{e}_z}{k} = \frac{\text{sgn}(k_z) k \mathbf{e}_z}{k} = \text{sgn}(k_z) \mathbf{e}_z$
- Phase of the plane wave /  
Phase der ebenen Welle  $k \hat{\mathbf{k}} \cdot \mathbf{R} = k \text{sgn}(k_z) \mathbf{e}_z \cdot (x \mathbf{e}_x + y \mathbf{e}_y + z \mathbf{e}_z) = k \text{sgn}(k_z) z \mathbf{e}_z \cdot \mathbf{e}_z = k \text{sgn}(k_z) z$

**Derivation of the Stability Condition for the 1-D FD Scheme of 2nd Order /  
Ableitung der Stabilitätsbedingung für das 1D-FD-Schema 2ter Ordnung**

Insert discrete plane wave / Setze die diskrete ebene Welle

$$\hat{E}_x^{(n_z, n_t)} = \hat{E}_{0x}(\omega_0, \hat{\mathbf{k}}) \underbrace{e^{jk\Delta z}}_{=\exp(n_z)} e^{-j\omega_0 n_t \Delta t} = \hat{E}_{0x}(\omega_0, \hat{\mathbf{k}}) \exp(n_z) e^{-j\omega_0 n_t \Delta t}$$

into the FD scheme / in das FD-Schema ein

$$\hat{E}_x^{(n_z, n_t+1)} = 2\hat{E}_x^{(n_z, n_t)} - \hat{E}_x^{(n_z, n_t-1)} + (\Delta t)^2 \left[ \hat{E}_x^{(n_z+1, n_t)} - 2\hat{E}_x^{(n_z, n_t)} + \hat{E}_x^{(n_z-1, n_t)} \right]$$

with / mit

$$\hat{E}_x^{(n_z, n_t+1)} = \hat{E}_{0x}(\omega_0, \hat{\mathbf{k}}) \underbrace{e^{jk\Delta z}}_{=\exp(n_z)} e^{-j\omega_0 (n_t+1)\Delta t}$$

$$= \hat{E}_{0x}(\omega_0, \hat{\mathbf{k}}) \exp(n_z) e^{-j\omega_0 (n_t+1)\Delta t}$$

$$\hat{E}_x^{(n_z, n_t)} = \hat{E}_{0x}(\omega_0, \hat{\mathbf{k}}) \exp(n_z) e^{-j\omega_0 n_t \Delta t}$$

$$\hat{E}_x^{(n_z, n_t-1)} = \hat{E}_{0x}(\omega_0, \hat{\mathbf{k}}) \exp(n_z) e^{-j\omega_0 (n_t-1)\Delta t}$$

$$\hat{E}_x^{(n_z+1, n_t)} = \hat{E}_{0x}(\omega_0, \hat{\mathbf{k}}) \exp(n_z) e^{jk\Delta z} e^{-j\omega_0 n_t \Delta t}$$

$$\hat{E}_x^{(n_z, n_t)} = \hat{E}_{0x}(\omega_0, \hat{\mathbf{k}}) \exp(n_z) e^{-j\omega_0 n_t \Delta t}$$

$$\hat{E}_x^{(n_z-1, n_t)} = \hat{E}_{0x}(\omega_0, \hat{\mathbf{k}}) \exp(n_z) e^{-jk\Delta z} e^{-j\omega_0 n_t \Delta t}$$

it follows / folgt

$$\begin{aligned} \hat{E}_{0x}(\omega_0, \hat{\mathbf{k}}) e^{-j\omega_0 (n_t+1)\Delta t} &= 2\hat{E}_{0x}(\omega_0, \hat{\mathbf{k}}) e^{-j\omega_0 n_t \Delta t} - \hat{E}_{0x}(\omega_0, \hat{\mathbf{k}}) e^{-j\omega_0 (n_t-1)\Delta t} \\ &\quad + (\Delta t)^2 \left[ \hat{E}_{0x}(\omega_0, \hat{\mathbf{k}}) e^{jk\Delta z} - 2\hat{E}_{0x}(\omega_0, \hat{\mathbf{k}}) + \hat{E}_{0x}(\omega_0, \hat{\mathbf{k}}) e^{-jk\Delta z} \right] e^{-j\omega_0 n_t \Delta t} \\ &= 2 \left[ 1 - (\Delta t)^2 \right] \hat{E}_{0x}(\omega_0, \hat{\mathbf{k}}) e^{-j\omega_0 n_t \Delta t} - \hat{E}_{0x}(\omega_0, \hat{\mathbf{k}}) e^{-j\omega_0 (n_t-1)\Delta t} \\ &\quad + (\Delta t)^2 \left[ \hat{E}_{0x}(\omega_0, \hat{\mathbf{k}}) e^{jk\Delta z} + \hat{E}_{0x}(\omega_0, \hat{\mathbf{k}}) e^{-jk\Delta z} \right] e^{-j\omega_0 n_t \Delta t} \end{aligned}$$

**Derivation of the Stability Condition for the 1-D FD Scheme of 2nd Order /  
Ableitung der Stabilitätsbedingung für das 1D-FD-Schema 2ter Ordnung**

$$\begin{aligned} \hat{E}_{0x}(\omega_0, \hat{\mathbf{k}}) e^{-j\omega_0 (n_t+1)\Delta t} &= 2 \left[ 1 - (\Delta t)^2 \right] \hat{E}_{0x}(\omega_0, \hat{\mathbf{k}}) e^{-j\omega_0 n_t \Delta t} - \hat{E}_{0x}(\omega_0, \hat{\mathbf{k}}) e^{-j\omega_0 (n_t-1)\Delta t} \\ &\quad + (\Delta t)^2 \left[ \hat{E}_{0x}(\omega_0, \hat{\mathbf{k}}) e^{jk\Delta z} + \hat{E}_{0x}(\omega_0, \hat{\mathbf{k}}) e^{-jk\Delta z} \right] e^{-j\omega_0 n_t \Delta t} \end{aligned}$$

$$= 2 \left[ 1 - (\Delta t)^2 \right] \hat{E}_{0x}(\omega_0, \hat{\mathbf{k}}) e^{-j\omega_0 n_t \Delta t} - \hat{E}_{0x}(\omega_0, \hat{\mathbf{k}}) e^{-j\omega_0 (n_t-1)\Delta t}$$

$$+ (\Delta t)^2 \underbrace{\left[ e^{jk\Delta z} + e^{-jk\Delta z} \right]}_{=2\cos(k\Delta z)} \hat{E}_{0x}(\omega_0, \hat{\mathbf{k}}) e^{-j\omega_0 n_t \Delta t}$$

$$= 2 \left[ 1 - (\Delta t)^2 \right] \hat{E}_{0x}(\omega_0, \hat{\mathbf{k}}) e^{-j\omega_0 n_t \Delta t} - \hat{E}_{0x}(\omega_0, \hat{\mathbf{k}}) e^{-j\omega_0 (n_t-1)\Delta t}$$

$$+ (\Delta t)^2 2\cos(k\Delta z) \hat{E}_{0x}(\omega_0, \hat{\mathbf{k}}) e^{-j\omega_0 n_t \Delta t}$$

$$\hat{E}_{0x}(\omega_0, \hat{\mathbf{k}}) e^{-j\omega_0 (n_t+1)\Delta t} = 2 \left\{ 1 + (\Delta t)^2 [\cos(k\Delta z) - 1] \right\} \hat{E}_{0x}(\omega_0, \hat{\mathbf{k}}) e^{-j\omega_0 n_t \Delta t} - \hat{E}_{0x}(\omega_0, \hat{\mathbf{k}}) e^{-j\omega_0 (n_t-1)\Delta t}$$

$$2\sin^2\left(\frac{\alpha}{2}\right) = 1 - \cos\alpha \quad \rightarrow \quad -2\sin^2\left(\frac{\alpha}{2}\right) = \cos\alpha - 1$$

$$\begin{aligned} \hat{E}_{0x}(\omega_0, \hat{\mathbf{k}}) e^{-j\omega_0 (n_t+1)\Delta t} &= 2 \left\{ 1 - 2(\Delta t)^2 \sin^2\left(\frac{k\Delta z}{2}\right) \right\} \hat{E}_{0x}(\omega_0, \hat{\mathbf{k}}) e^{-j\omega_0 n_t \Delta t} - \hat{E}_{0x}(\omega_0, \hat{\mathbf{k}}) e^{-j\omega_0 (n_t-1)\Delta t} \\ &= -\hat{E}_{0x}(\omega_0, \hat{\mathbf{k}}) e^{-j\omega_0 (n_t-1)\Delta t} + 2 \left\{ 1 - 2(\Delta t)^2 \sin^2\left(\frac{k\Delta z}{2}\right) \right\} \hat{E}_{0x}(\omega_0, \hat{\mathbf{k}}) e^{-j\omega_0 n_t \Delta t} \end{aligned}$$



**Derivation of the Stability Condition for the 1-D FD Scheme of 2nd Order /  
Ableitung der Stabilitätsbedingung für das 1D-FD-Schema 2ter Ordnung**

$$\begin{aligned}\hat{E}_{0x}(\omega_0, \hat{\mathbf{k}}) e^{-j\omega_0(n+1)\Delta t} &= 2 \left\{ 1 - 2(\hat{\Delta t})^2 \sin^2 \left( \frac{k\Delta z}{2} \right) \right\} \hat{E}_{0x}(\omega_0, \hat{\mathbf{k}}) e^{-j\omega_0 n \Delta t} - \hat{E}_{0x}(\omega_0, \hat{\mathbf{k}}) e^{-j\omega_0(n-1)\Delta t} \\ &= -\hat{E}_{0x}(\omega_0, \hat{\mathbf{k}}) e^{-j\omega_0(n-1)\Delta t} + 2 \left\{ 1 - 2(\hat{\Delta t})^2 \sin^2 \left( \frac{k\Delta z}{2} \right) \right\} \hat{E}_{0x}(\omega_0, \hat{\mathbf{k}}) e^{-j\omega_0 n \Delta t}\end{aligned}$$

Define / Definiere

$$\begin{aligned}U^{(n)} &= \hat{E}_{0x}(\omega_0, \hat{\mathbf{k}}) e^{-j\omega_0 n \Delta t} \\ V^{(n)} &= U^{(n-1)} \\ &= \hat{E}_{0x}(\omega_0, \hat{\mathbf{k}}) e^{-j\omega_0(n-1)\Delta t}\end{aligned}$$

which yields for the above equation / womit wir für die obere Gleichung erhalten

$$\begin{aligned}U^{(n+1)} &= -U^{(n-1)} + 2 \left\{ 1 - 2(\hat{\Delta t})^2 \sin^2 \left( \frac{k\Delta z}{2} \right) \right\} U^{(n)} \\ &= -V^{(n)} + 2 \left\{ 1 - 2(\hat{\Delta t})^2 \sin^2 \left( \frac{k\Delta z}{2} \right) \right\} U^{(n)}\end{aligned}$$

**Derivation of the Stability Condition for the 1-D FD Scheme of 2nd Order /  
Ableitung der Stabilitätsbedingung für das 1D-FD-Schema 2ter Ordnung**

Define a new algebraic vector / Definiere einen neuen algebraischen Vektor

$$\{\mathbf{W}\}^{(n)} = \begin{Bmatrix} U^{(n)} \\ V^{(n)} \end{Bmatrix}$$

$$\begin{aligned}\begin{Bmatrix} U^{(n+1)} \\ V^{(n+1)} \end{Bmatrix} &= \begin{bmatrix} 2 \left\{ 1 - 2(\hat{\Delta t})^2 \sin^2 \left( \frac{k\Delta z}{2} \right) \right\} & -1 \\ 1 & 0 \end{bmatrix} \begin{Bmatrix} U^{(n)} \\ V^{(n)} \end{Bmatrix} \\ \underbrace{\begin{Bmatrix} U^{(n+1)} \\ V^{(n+1)} \end{Bmatrix}}_{=\{\mathbf{W}\}^{(n+1)}} &= \underbrace{\begin{bmatrix} 2 \left\{ 1 - 2(\hat{\Delta t})^2 \sin^2 \left( \frac{k\Delta z}{2} \right) \right\} & -1 \\ 1 & 0 \end{bmatrix}}_{=[\mathbf{G}]_{\text{ID}}^{\text{FD}}} \underbrace{\begin{Bmatrix} U^{(n)} \\ V^{(n)} \end{Bmatrix}}_{=\{\mathbf{W}\}^{(n)}} \\ \{\mathbf{W}\}^{(n+1)} &= [\mathbf{G}]_{\text{ID}}^{\text{FD}} \{\mathbf{W}\}^{(n)} \quad \quad \quad [\mathbf{G}]_{\text{ID}}^{\text{FD}} : \text{Amplification matrix / Verstärkungsmatrix}\end{aligned}$$

$$\det \{ [\mathbf{G}]_{\text{ID}}^{\text{FD}} - v[\mathbf{I}] \} = 0 \quad v_n \left( [\mathbf{G}]_{\text{ID}}^{\text{FD}} \right) : n\text{th eigenvalue of the matrix } [\mathbf{G}]_{\text{ID}}^{\text{FD}} \\ n\text{-ter Eigenwert der Matrix } [\mathbf{G}]_{\text{ID}}^{\text{FD}}$$

$$\begin{aligned}\det \{ [\mathbf{G}]_{\text{ID}}^{\text{FD}} - v[\mathbf{I}] \} &= \begin{vmatrix} 2 \left\{ 1 - 2(\hat{\Delta t})^2 \sin^2 \left( \frac{k\Delta z}{2} \right) \right\} - v & -1 \\ 1 & -v \end{vmatrix} \\ &= v^2 - 2v \left\{ 1 - 2(\hat{\Delta t})^2 \sin^2 \left( \frac{k\Delta z}{2} \right) \right\} + 1\end{aligned}$$

Characteristic polynomial /  
Charakteristisches Polynom

**Derivation of the Stability Condition for the 1-D FD Scheme of 2nd Order /  
Ableitung der Stabilitätsbedingung für das 1D-FD-Schema 2ter Ordnung**

$$v^2 - 2v \left[ 1 - 2(\widehat{\Delta t})^2 \sin^2 \left( \frac{k\Delta z}{2} \right) \right] + \left[ 1 - 2(\widehat{\Delta t})^2 \sin^2 \left( \frac{k\Delta z}{2} \right) \right]^2 = \left[ 1 - 2(\widehat{\Delta t})^2 \sin^2 \left( \frac{k\Delta z}{2} \right) \right]^2 - 1$$

$$\left\{ v - \left[ 1 - 2(\widehat{\Delta t})^2 \sin^2 \left( \frac{k\Delta z}{2} \right) \right] \right\}^2 = \left[ 1 - 2(\widehat{\Delta t})^2 \sin^2 \left( \frac{k\Delta z}{2} \right) \right]^2 - 1$$

Eigenvalues of the amplification matrix /  
Eigenwerte der Verstärkungsmatrix

$$v_{1/2} = \underbrace{\left\{ 1 - 2(\widehat{\Delta t})^2 \sin^2 \left( \frac{k\Delta z}{2} \right) \right\}}_{=a} \pm \sqrt{\underbrace{\left\{ 1 - 2(\widehat{\Delta t})^2 \sin^2 \left( \frac{k\Delta z}{2} \right) \right\}^2}_{=a^2} - 1}$$

$$= a \pm \sqrt{a^2 - 1}$$

if  $a^2 \leq 1$  / falls  $a^2 \leq 1$

$$= a \pm j\sqrt{1 - a^2}$$

**Derivation of the Stability Condition for the 1-D FD Scheme of 2nd Order /  
Ableitung der Stabilitätsbedingung für das 1D-FD-Schema 2ter Ordnung**

$$v_{1/2} = \underbrace{\left\{ 1 - 2(\widehat{\Delta t})^2 \sin^2 \left( \frac{k\Delta z}{2} \right) \right\}}_{=a} \pm \sqrt{\underbrace{\left\{ 1 - 2(\widehat{\Delta t})^2 \sin^2 \left( \frac{k\Delta z}{2} \right) \right\}^2}_{=a^2} - 1}$$

$$= a \pm \sqrt{a^2 - 1}$$

$$= a \pm j\sqrt{1 - a^2} \quad \text{if } a^2 \leq 1 / \text{falls } a^2 \leq 1$$

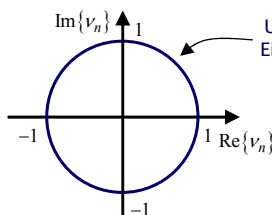
$$v_n = \text{Re}\{v_n\} + j\text{Im}\{v_n\} \quad n = 1, 2$$

$$|v_{1/2}| = \left| a \pm j\sqrt{1 - a^2} \right| = a^2 + \left( \sqrt{1 - a^2} \right)^2 = a^2 + 1 - a^2 = 1$$

Spectral radius /  
Spektraler Radius

$$\rho([\mathbf{G}]_{1D}^{\text{FD}}) \leq 1$$

This means for, that all eigenvalues  $a^2 \leq 1$  are on the unit circle in the complex plane. /  
Dies bedeutet, dass alle Eigenwerte für  $a^2 \leq 1$  auf dem Einheitskreis in der komplexen Ebene liegen.



**Derivation of the Stability Condition for the 1-D FD Scheme of 2nd Order /  
Ableitung der Stabilitätsbedingung für das 1D-FD-Schema 2ter Ordnung**

$$v_{1/2} = \underbrace{\left\{ 1 - 2(\widehat{\Delta t})^2 \sin^2\left(\frac{k\Delta x}{2}\right) \right\}}_{=a} \pm j \sqrt{\underbrace{\left\{ 1 - 2(\widehat{\Delta t})^2 \sin^2\left(\frac{k\Delta x}{2}\right) \right\}^2}_{=a^2}}$$

$$= a \pm j\sqrt{1-a^2} \quad \text{if } a^2 \leq 1 / \text{falls } a^2 \leq 1$$

$$a^2 \leq 1$$

$$\left\{ 1 - 2(\widehat{\Delta t})^2 \sin^2\left(\frac{k\Delta x}{2}\right) \right\}^2 \leq 1$$

$$1 - 4(\widehat{\Delta t})^2 \sin^2\left(\frac{k\Delta x}{2}\right) + 4(\widehat{\Delta t})^4 \sin^4\left(\frac{k\Delta x}{2}\right) \leq 1$$

$$-4(\widehat{\Delta t})^2 \sin^2\left(\frac{k\Delta x}{2}\right) \left[ 1 - (\widehat{\Delta t})^2 \sin^2\left(\frac{k\Delta x}{2}\right) \right] \leq 0$$

$$1 - (\widehat{\Delta t})^2 \sin^2\left(\frac{k\Delta x}{2}\right) \leq 0$$

$$(\widehat{\Delta t})^2 \sin^2\left(\frac{k\Delta x}{2}\right) \leq 1$$

$$(\widehat{\Delta t})^2 \leq 1 \quad \text{because / weil } \max\left\{ \sin^2\left(\frac{k\Delta x}{2}\right) \right\} = 1$$

$$\widehat{\Delta t} \leq 1$$

**Derivation of the Stability Condition for the 1-D FD Scheme of 2nd Order /  
Ableitung der Stabilitätsbedingung für das 1D-FD-Schema 2ter Ordnung**

1-D Stability Condition for an FD algorithm of 2nd order in space and time- CFL-Condition /  
1D-Stabilitätsbedingung für einen FD-Algorithmus zweiter Ordnung in Raum und Zeit- CFL-  
Bedingung

$$\boxed{1\text{-D} / 1\text{D}: \Delta t \leq \Delta t_{\max} = \frac{\Delta x}{c} \quad \widehat{\Delta t} \leq 1}$$

2-D and 3-D Stability Condition for an FD algorithm of 2nd order in space and time- CFL-  
Condition /  
2D- und 3D- Stabilitätsbedingung für einen FD-Algorithmus zweiter Ordnung in Raum und Zeit-  
CFL-Bedingung

$$\boxed{2\text{-D} / 2\text{D}: \Delta t \leq \Delta t_{\max} = \frac{1}{\sqrt{2}} \frac{\Delta x}{c} \quad \widehat{\Delta t} \leq \frac{1}{\sqrt{2}} \approx 0.707}$$

$$3\text{-D} / 3\text{D}: \Delta t \leq \Delta t_{\max} = \frac{1}{\sqrt{3}} \frac{\Delta x}{c} \quad \widehat{\Delta t} \leq \frac{1}{\sqrt{3}} \approx 0.577$$

$$\widehat{\Delta t} = \frac{\Delta t}{\Delta t_{\text{ref}}} : \quad \text{Courant number / Courant - Zahl} \quad \Delta t_{\text{ref}} = \frac{\Delta x}{c}$$

**Derivation of the Stability Condition for the 1-D FD Scheme of 2nd Order /  
Ableitung der Stabilitätsbedingung für das 1D-FD-Schema 2ter Ordnung**

$$v_{1/2}(\hat{\Delta t}) = a \pm \sqrt{a^2 - 1} \quad \text{if } a^2 \geq 1 / \text{falls } a^2 \geq 1$$

$$= a \pm j\sqrt{1 - a^2} \quad \text{if } a^2 \leq 1 / \text{falls } a^2 \leq 1 \quad \text{with } a = \left\{ 1 - 2(\hat{\Delta t})^2 \sin^2\left(\frac{k\Delta z}{2}\right) \right\}$$

$$a^2 \leq 1: v_{1/2} = a \pm j\sqrt{1 - a^2}$$

$$v_1 = a + j\sqrt{1 - a^2} \quad v_2 = a - j\sqrt{1 - a^2}$$

$$|v_1| = 1 \quad |v_2| = 1$$

Spectral radius /  
Spektraler Radius  $\rho([G]_{1D}^{FD}) \leq 1$

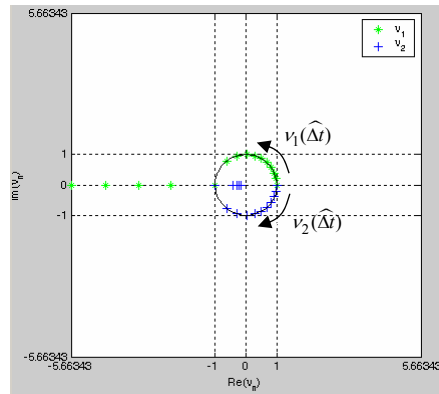
$$a^2 > 1: v_{1/2} = a \pm \sqrt{a^2 - 1}$$

$$v_1 = a + \sqrt{a^2 - 1} \quad v_2 = a - \sqrt{a^2 - 1}$$

$$\lim_{a \rightarrow \infty} |v_1| \rightarrow \infty \quad \lim_{a \rightarrow \infty} |v_2| \rightarrow 0$$

Spectral radius /  
Spektraler Radius  $\rho([G]_{1D}^{FD}) \geq 1$

$v_{1/2}(\hat{\Delta t})$  as a function of  $(\hat{\Delta t})$   
als Funktion von  $(\hat{\Delta t})$



**Derivation of the Stability Condition for the 1-D FD Scheme of 2nd Order /  
Ableitung der Stabilitätsbedingung für das 1D-FD-Schema 2ter Ordnung**

Eigenvalues / Eigenwerte

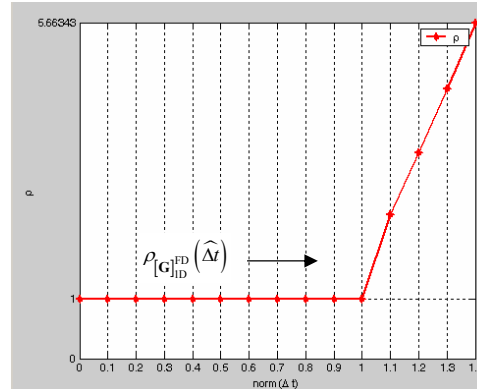
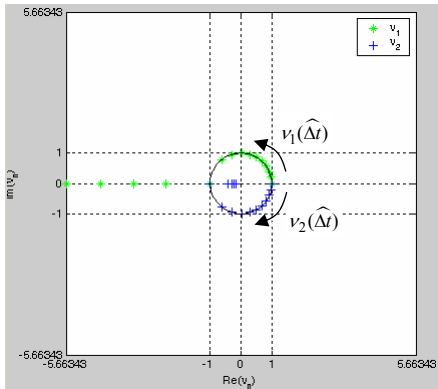
$$v_{1/2}(\hat{\Delta t}) = a \pm \sqrt{a^2 - 1} \quad \text{if } a^2 \geq 1 / \text{falls } a^2 \geq 1$$

$$= a \pm j\sqrt{1 - a^2} \quad \text{if } a^2 \leq 1 / \text{falls } a^2 \leq 1$$

$$\text{with } a = \left\{ 1 - 2(\hat{\Delta t})^2 \sin^2\left(\frac{k\Delta z}{2}\right) \right\}$$

Spectral radius / Spektraler Radius

$$\rho_{[G]_{1D}^{FD}}(\hat{\Delta t})$$



**End of Lecture 4 /  
Ende der 4. Vorlesung**