# Numerical Methods of Electromagnetic Field Theory I (NFT I) Numerische Methoden der Elektromagnetischen Feldtheorie I (NFT I) /

## 8th Lecture / 8. Vorlesung


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# 3-D FDTD – Derivation of the Discrete Equations / 3D-FDTD – Ableitung der diskreten Gleichungen

The first two Maxwell's Equations are in differential form /
Die ersten beiden Maxwellschen Gleichungen lauten in Differentialform:

$$\frac{\partial}{\partial t}\underline{\mathbf{B}}(\underline{\mathbf{R}},t) = -\nabla\times\underline{\mathbf{E}}(\underline{\mathbf{R}},t) - \underline{\mathbf{J}}_{\mathrm{m}}(\underline{\mathbf{R}},t)$$

$$\frac{\partial}{\partial t}\underline{\mathbf{D}}(\underline{\mathbf{R}},t) = \nabla\times\underline{\mathbf{H}}(\underline{\mathbf{R}},t) - \underline{\mathbf{J}}_{\mathrm{e}}(\underline{\mathbf{R}},t)$$

In Cartesian Coordinates we find for the Curl operator applied to E and H /
Im Kartesischen Koordinatensystem finden wir für den Rotationsoperator angewendet auf E und H:

$$\nabla\times\underline{\mathbf{E}}(\underline{\mathbf{R}},t) = \begin{vmatrix} \underline{\mathbf{e}}_x & \underline{\mathbf{e}}_y & \underline{\mathbf{e}}_y \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x(\underline{\mathbf{R}},t) & E_y(\underline{\mathbf{R}},t) & E_z(\underline{\mathbf{R}},t) \end{vmatrix}$$

$$= \left[\frac{\partial E_z(\underline{\mathbf{R}},t)}{\partial y} - \frac{\partial E_y(\underline{\mathbf{R}},t)}{\partial z}\right]\underline{\mathbf{e}}_x + \left[\frac{\partial E_x(\underline{\mathbf{R}},t)}{\partial z} - \frac{\partial E_z(\underline{\mathbf{R}},t)}{\partial x}\right]\underline{\mathbf{e}}_y + \left[\frac{\partial E_y(\underline{\mathbf{R}},t)}{\partial x} - \frac{\partial E_x(\underline{\mathbf{R}},t)}{\partial y}\right]\underline{\mathbf{e}}_z$$

$$\nabla\times\underline{\mathbf{H}}(\underline{\mathbf{R}},t) = \begin{vmatrix} \underline{\mathbf{e}}_x & \underline{\mathbf{e}}_y & \underline{\mathbf{e}}_y \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x(\underline{\mathbf{R}},t) & H_y(\underline{\mathbf{R}},t) & H_z(\underline{\mathbf{R}},t) \end{vmatrix}$$

$$= \left[\frac{\partial H_z(\underline{\mathbf{R}},t)}{\partial y} - \frac{\partial H_y(\underline{\mathbf{R}},t)}{\partial z}\right]\underline{\mathbf{e}}_x + \left[\frac{\partial H_x(\underline{\mathbf{R}},t)}{\partial z} - \frac{\partial H_z(\underline{\mathbf{R}},t)}{\partial x}\right]\underline{\mathbf{e}}_y + \left[\frac{\partial H_y(\underline{\mathbf{R}},t)}{\partial x} - \frac{\partial H_x(\underline{\mathbf{R}},t)}{\partial y}\right]\underline{\mathbf{e}}_z$$

# 3-D FDTD – Derivation of the Discrete Equations / 3D-FDTD – Ableitung der diskreten Gleichungen

If we insert the last expressions into the first two Maxwell's equations are in differential form read /
Wenn wir die letzten Ausdrücke in the ersten beiden Maxwellschen Gleichungen in Differentialform einsetzen, erhalten wir:

$$\frac{\partial}{\partial t}\underline{\mathbf{B}}(\underline{\mathbf{R}},t) = -\nabla\times\underline{\mathbf{E}}(\underline{\mathbf{R}},t) - \underline{\mathbf{J}}_{\mathrm{m}}(\underline{\mathbf{R}},t)$$

$$\frac{\partial}{\partial t}\left[B_x(\underline{\mathbf{R}},t)\underline{\mathbf{e}}_x + B_y(\underline{\mathbf{R}},t)\underline{\mathbf{e}}_y + B_z(\underline{\mathbf{R}},t)\underline{\mathbf{e}}_z\right]$$
$$= -\left\{\left[\frac{\partial E_z(\underline{\mathbf{R}},t)}{\partial y} - \frac{\partial E_y(\underline{\mathbf{R}},t)}{\partial z}\right]\underline{\mathbf{e}}_x + \left[\frac{\partial E_x(\underline{\mathbf{R}},t)}{\partial z} - \frac{\partial E_z(\underline{\mathbf{R}},t)}{\partial x}\right]\underline{\mathbf{e}}_y + \left[\frac{\partial E_y(\underline{\mathbf{R}},t)}{\partial x} - \frac{\partial E_x(\underline{\mathbf{R}},t)}{\partial y}\right]\underline{\mathbf{e}}_z\right\}$$
$$-\left[J_{\mathrm{m}x}(\underline{\mathbf{R}},t)\underline{\mathbf{e}}_x + J_{\mathrm{m}y}(\underline{\mathbf{R}},t)\underline{\mathbf{e}}_y + J_{\mathrm{m}z}(\underline{\mathbf{R}},t)\underline{\mathbf{e}}_z\right]$$

$$\frac{\partial}{\partial t}\underline{\mathbf{D}}(\underline{\mathbf{R}},t) = \nabla\times\underline{\mathbf{H}}(\underline{\mathbf{R}},t) - \underline{\mathbf{J}}_{\mathrm{e}}(\underline{\mathbf{R}},t)$$

$$\frac{\partial}{\partial t}\left[D_x(\underline{\mathbf{R}},t)\underline{\mathbf{e}}_x + D_y(\underline{\mathbf{R}},t)\underline{\mathbf{e}}_y + D_z(\underline{\mathbf{R}},t)\underline{\mathbf{e}}_z\right]$$
$$= \left[\frac{\partial H_z(\underline{\mathbf{R}},t)}{\partial y} - \frac{\partial H_y(\underline{\mathbf{R}},t)}{\partial z}\right]\underline{\mathbf{e}}_x + \left[\frac{\partial H_x(\underline{\mathbf{R}},t)}{\partial z} - \frac{\partial H_z(\underline{\mathbf{R}},t)}{\partial x}\right]\underline{\mathbf{e}}_y + \left[\frac{\partial H_y(\underline{\mathbf{R}},t)}{\partial x} - \frac{\partial H_x(\underline{\mathbf{R}},t)}{\partial y}\right]\underline{\mathbf{e}}_z$$
$$-\left[J_{\mathrm{e}x}(\underline{\mathbf{R}},t)\underline{\mathbf{e}}_x + J_{\mathrm{e}y}(\underline{\mathbf{R}},t)\underline{\mathbf{e}}_y + J_{\mathrm{e}z}(\underline{\mathbf{R}},t)\underline{\mathbf{e}}_z\right]$$

Six decoupled scalar equations! /
Sechs entkoppelte skalare Gleichungen!

# 3-D FDTD – Derivation of the Discrete Equations / 3D-FDTD – Ableitung der diskreten Gleichungen

If we insert the last expressions into the first two Maxwell's equations are in differential form we read / Wenn wir die letzten Ausdrücke in die ersten beiden Maxwellschen Gleichungen in Differentialform einsetzen, erhalten wir:

$$\frac{\partial}{\partial t} B_x(\underline{\mathbf{R}},t) = -\left[\frac{\partial E_z(\underline{\mathbf{R}},t)}{\partial y} - \frac{\partial E_y(\underline{\mathbf{R}},t)}{\partial z}\right] - J_{\mathrm{m}x}(\underline{\mathbf{R}},t)$$

$$\frac{\partial}{\partial t} B_y(\underline{\mathbf{R}},t) = -\left[\frac{\partial E_x(\underline{\mathbf{R}},t)}{\partial z} - \frac{\partial E_z(\underline{\mathbf{R}},t)}{\partial x}\right] - J_{\mathrm{m}y}(\underline{\mathbf{R}},t)$$

$$\frac{\partial}{\partial t} B_z(\underline{\mathbf{R}},t) = -\left[\frac{\partial E_y(\underline{\mathbf{R}},t)}{\partial x} - \frac{\partial E_x(\underline{\mathbf{R}},t)}{\partial y}\right] - J_{\mathrm{m}z}(\underline{\mathbf{R}},t)$$

$$\frac{\partial}{\partial t} D_x(\underline{\mathbf{R}},t) = \left[\frac{\partial H_z(\underline{\mathbf{R}},t)}{\partial y} - \frac{\partial H_y(\underline{\mathbf{R}},t)}{\partial z}\right] - J_{\mathrm{e}x}(\underline{\mathbf{R}},t)$$

$$\frac{\partial}{\partial t} D_y(\underline{\mathbf{R}},t) = \left[\frac{\partial H_x(\underline{\mathbf{R}},t)}{\partial z} - \frac{\partial H_z(\underline{\mathbf{R}},t)}{\partial x}\right] - J_{\mathrm{e}y}(\underline{\mathbf{R}},t)$$

$$\frac{\partial}{\partial t} D_z(\underline{\mathbf{R}},t) = \left[\frac{\partial H_y(\underline{\mathbf{R}},t)}{\partial x} - \frac{\partial H_x(\underline{\mathbf{R}},t)}{\partial y}\right] - J_{\mathrm{e}z}(\underline{\mathbf{R}},t)$$

# 3-D FDTD – Derivation of the Discrete Equations / 3D-FDTD – Ableitung der diskreten Gleichungen

Constitutive equation for homogeneous isotropic materials / Konstituierende Gleichungen für homogene isotrope Materialien:

$$B_x(\underline{\mathbf{R}},t) = \mu H_x(\underline{\mathbf{R}},t) \qquad D_x(\underline{\mathbf{R}},t) = \varepsilon E_x(\underline{\mathbf{R}},t)$$
$$B_y(\underline{\mathbf{R}},t) = \mu H_y(\underline{\mathbf{R}},t) \qquad D_y(\underline{\mathbf{R}},t) = \varepsilon E_y(\underline{\mathbf{R}},t)$$
$$B_z(\underline{\mathbf{R}},t) = \mu H_z(\underline{\mathbf{R}},t) \qquad D_z(\underline{\mathbf{R}},t) = \varepsilon E_z(\underline{\mathbf{R}},t)$$

$$\frac{\partial}{\partial t}\mu H_x(\underline{\mathbf{R}},t) = -\left[\frac{\partial E_z(\underline{\mathbf{R}},t)}{\partial y} - \frac{\partial E_y(\underline{\mathbf{R}},t)}{\partial z}\right] - J_{\mathrm{m}x}(\underline{\mathbf{R}},t)$$

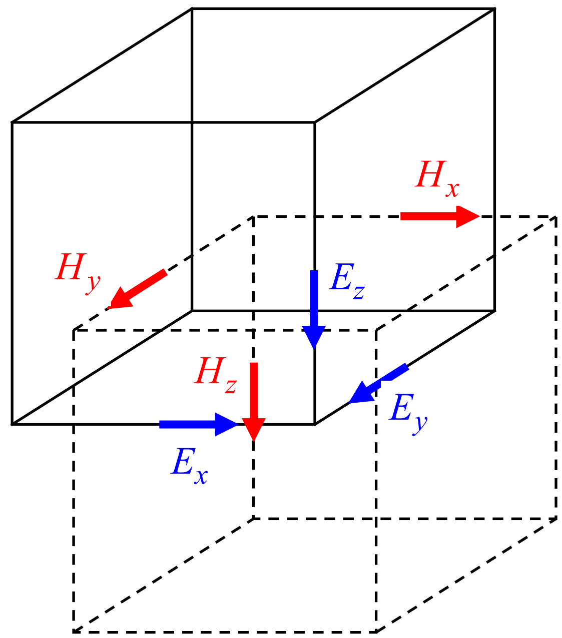
$$\frac{\partial}{\partial t}\mu H_y(\underline{\mathbf{R}},t) = -\left[\frac{\partial E_x(\underline{\mathbf{R}},t)}{\partial z} - \frac{\partial E_z(\underline{\mathbf{R}},t)}{\partial x}\right] - J_{\mathrm{m}y}(\underline{\mathbf{R}},t)$$

$$\frac{\partial}{\partial t}\mu H_z(\underline{\mathbf{R}},t) = -\left[\frac{\partial E_y(\underline{\mathbf{R}},t)}{\partial x} - \frac{\partial E_x(\underline{\mathbf{R}},t)}{\partial y}\right] - J_{\mathrm{m}z}(\underline{\mathbf{R}},t)$$

$$\frac{\partial}{\partial t}\varepsilon E_x(\underline{\mathbf{R}},t) = \left[\frac{\partial H_z(\underline{\mathbf{R}},t)}{\partial y} - \frac{\partial H_y(\underline{\mathbf{R}},t)}{\partial z}\right] - J_{\mathrm{e}x}(\underline{\mathbf{R}},t)$$

$$\frac{\partial}{\partial t}\varepsilon E_y(\underline{\mathbf{R}},t) = \left[\frac{\partial H_x(\underline{\mathbf{R}},t)}{\partial z} - \frac{\partial H_z(\underline{\mathbf{R}},t)}{\partial x}\right] - J_{\mathrm{e}y}(\underline{\mathbf{R}},t)$$

$$\frac{\partial}{\partial t}\varepsilon E_z(\underline{\mathbf{R}},t) = \left[\frac{\partial H_y(\underline{\mathbf{R}},t)}{\partial x} - \frac{\partial H_x(\underline{\mathbf{R}},t)}{\partial y}\right] - J_{\mathrm{e}z}(\underline{\mathbf{R}},t)$$




$$H_{x_i} = J_{\mathrm{m}x_i}, i = 1,2,3$$
$$E_{x_i} = J_{\mathrm{e}x_i}, i = 1,2,3$$

# 3-D FDTD – Derivation of the Discrete Equations / 3D-FDTD – Ableitung der diskreten Gleichungen

$$\frac{\partial}{\partial t} H_x(\underline{\mathbf{R}},t) = \dot{H}_x(\underline{\mathbf{R}},t)$$

$$\mu \dot{H}_x(\underline{\mathbf{R}},t) = -\left[\frac{\partial E_z(\underline{\mathbf{R}},t)}{\partial y} - \frac{\partial E_y(\underline{\mathbf{R}},t)}{\partial z}\right] - J_{\mathrm{mx}}(\underline{\mathbf{R}},t)$$

$$\mu \dot{H}_x(\underline{\mathbf{R}},t) = \dot{H}_x^{(m)}(t)$$

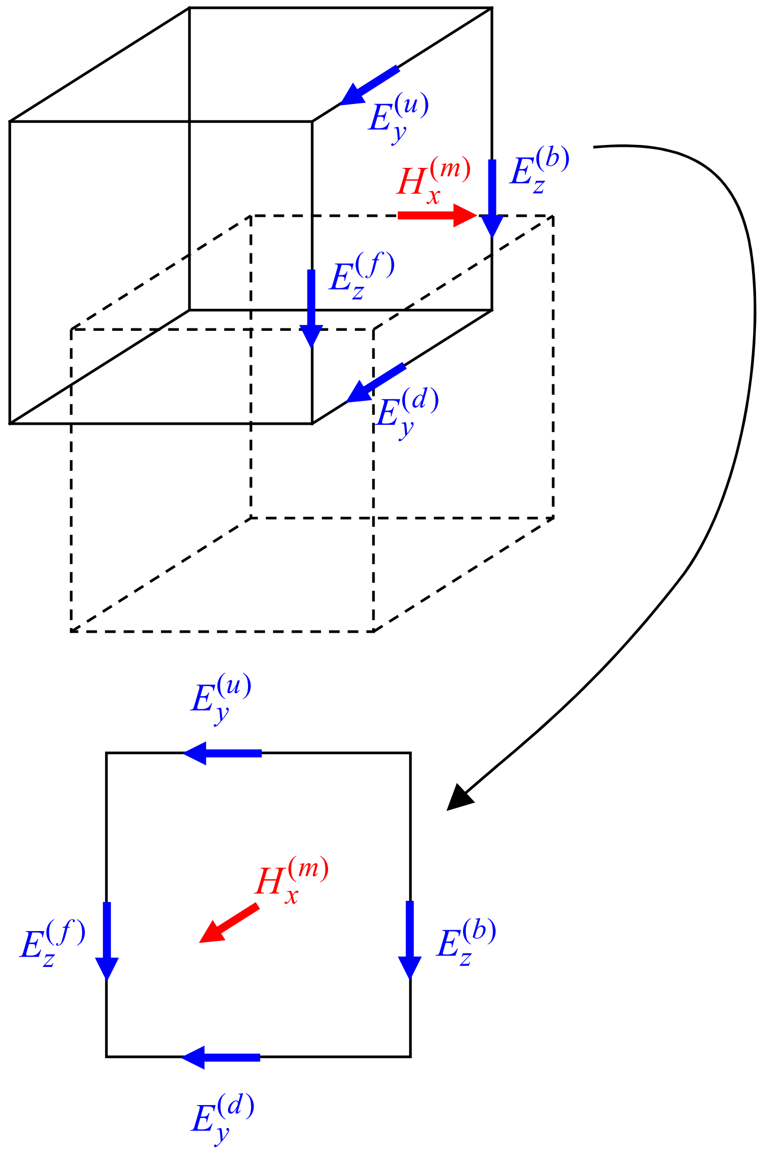
$$J_{\mathrm{mx}}(\underline{\mathbf{R}},t) = J_{\mathrm{mx}}^{(m)}(t)$$

$$\frac{\partial E_z(\underline{\mathbf{R}},t)}{\partial y} = \frac{E_z^{(f)}(t) - E_z^{(b)}(t)}{\Delta y} + \bigcirc\left[(\Delta y)^2\right]$$

$$\frac{\partial E_y(\underline{\mathbf{R}},t)}{\partial z} = \frac{E_y^{(d)}(t) - E_y^{(u)}(t)}{\Delta z} + \bigcirc\left[(\Delta z)^2\right]$$

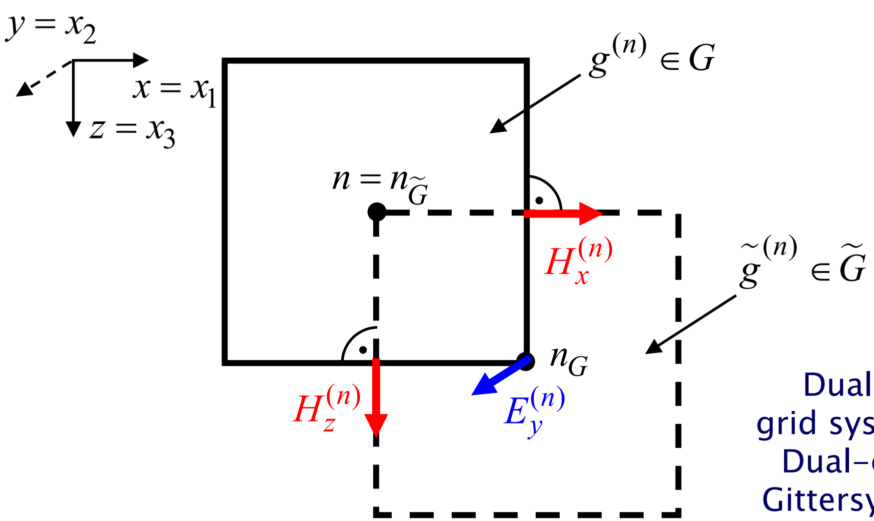
$$\mu \dot{H}_x^{(m)}(t) = \underbrace{-\frac{E_z^{(f)}(t) - E_z^{(b)}(t)}{\Delta y} + \frac{E_y^{(d)}(t) - E_y^{(u)}(t)}{\Delta z}} - J_{\mathrm{mx}}^{(m)}(t)$$

A part of the discrete curl operator / Ein Teil des diskreten Rotationsoperators

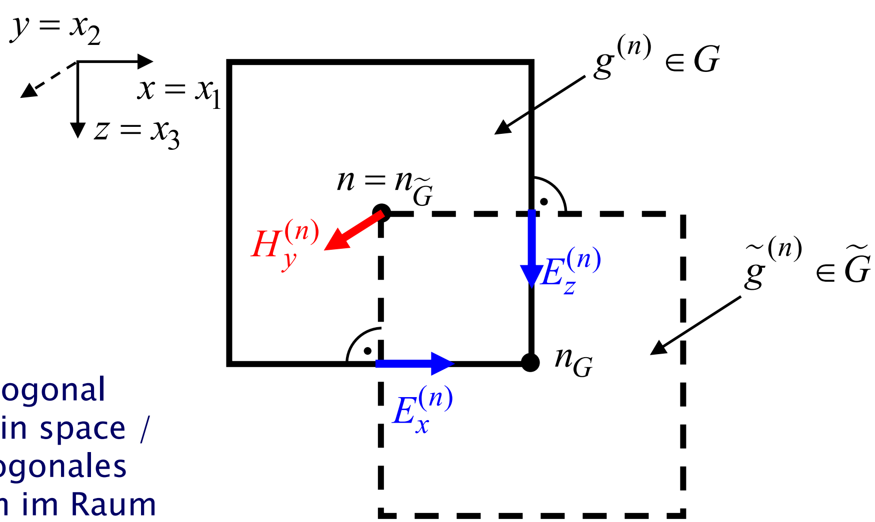


# 2-D EM Wave Propagation – 2-D FDTD – TM and TE Case / 2D EM Wellenausbreitung – 2D-FDTD – TM- und TE-Fall

## 2-D TM Case / 2D-TM-Fall




## 2-D TE Case / 2D-TE-Fall




Dual orthogonal grid system in space / Dual-orthogonales Gittersystem im Raum

$$G \perp \tilde{G}$$

$$\frac{\partial}{\partial t}\mu H_x(\underline{\mathbf{R}},t) = \frac{\partial E_y(\underline{\mathbf{R}},t)}{\partial z} - J_{\mathrm{mx}}(\underline{\mathbf{R}},t)$$

$$\frac{\partial}{\partial t}\mu H_z(\underline{\mathbf{R}},t) = -\frac{\partial E_y(\underline{\mathbf{R}},t)}{\partial x} - J_{\mathrm{mz}}(\underline{\mathbf{R}},t)$$

$$\frac{\partial}{\partial t}\varepsilon E_y(\underline{\mathbf{R}},t) = \left[\frac{\partial H_x(\underline{\mathbf{R}},t)}{\partial z} - \frac{\partial H_z(\underline{\mathbf{R}},t)}{\partial x}\right] - J_{\mathrm{ey}}(\underline{\mathbf{R}},t)$$

$$\underline{\mathbf{R}} = x\underline{\mathbf{e}}_x + z\underline{\mathbf{e}}_z$$

$$\frac{\partial}{\partial t}\mu H_y(\underline{\mathbf{R}},t) = -\left[\frac{\partial E_x(\underline{\mathbf{R}},t)}{\partial z} - \frac{\partial E_z(\underline{\mathbf{R}},t)}{\partial x}\right] - J_{\mathrm{my}}(\underline{\mathbf{R}},t)$$

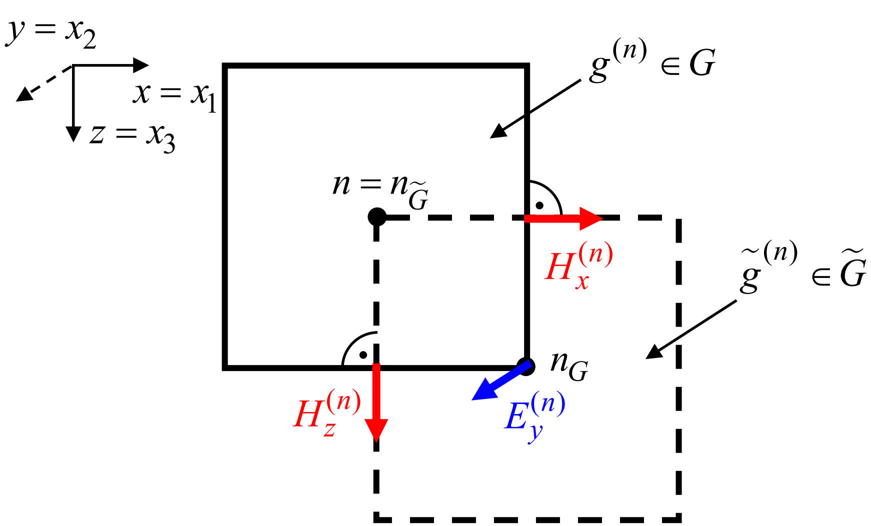
$$\frac{\partial}{\partial t}\varepsilon E_x(\underline{\mathbf{R}},t) = -\frac{\partial H_y(\underline{\mathbf{R}},t)}{\partial z} - J_{\mathrm{ex}}(\underline{\mathbf{R}},t)$$

$$\frac{\partial}{\partial t}\varepsilon E_z(\underline{\mathbf{R}},t) = \frac{\partial H_y(\underline{\mathbf{R}},t)}{\partial x} - J_{\mathrm{ez}}(\underline{\mathbf{R}},t)$$

$$\underline{\mathbf{R}} = x\underline{\mathbf{e}}_x + z\underline{\mathbf{e}}_z$$

# 2-D EM Wave Propagation – 2-D FDTD – TM Case/ 2D EM Wellenausbreitung – 2D-FDTD – TM-Fall

## 2-D TM Case / 2D-TM-Fall



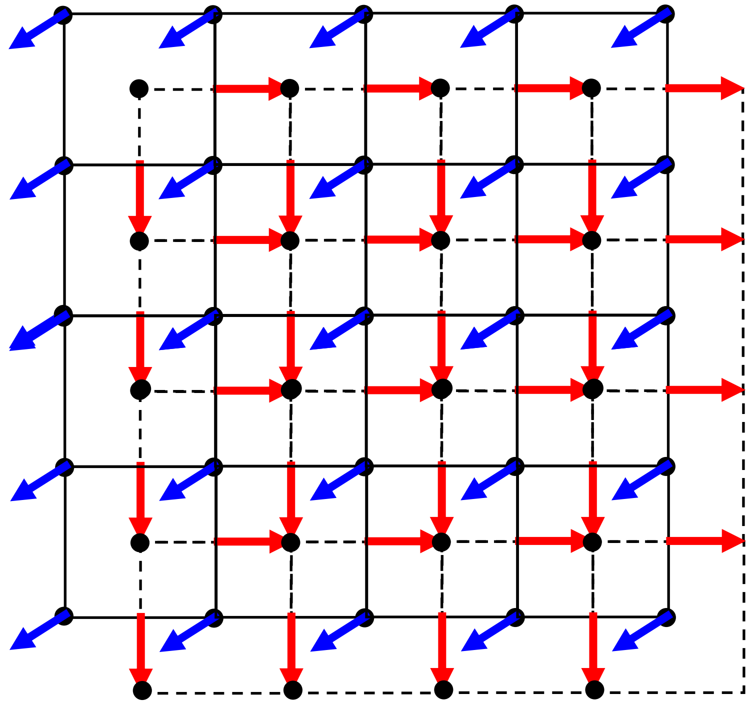

$$\frac{\partial}{\partial t}\mu H_x(\underline{\mathbf{R}},t) = \frac{\partial E_y(\underline{\mathbf{R}},t)}{\partial z} - J_{mx}(\underline{\mathbf{R}},t)$$

$$\frac{\partial}{\partial t}\mu H_z(\underline{\mathbf{R}},t) = -\frac{\partial E_y(\underline{\mathbf{R}},t)}{\partial x} - J_{mz}(\underline{\mathbf{R}},t)$$

$$\frac{\partial}{\partial t}\varepsilon E_y(\underline{\mathbf{R}},t) = \left[\frac{\partial H_x(\underline{\mathbf{R}},t)}{\partial z} - \frac{\partial H_z(\underline{\mathbf{R}},t)}{\partial x}\right] - J_{ey}(\underline{\mathbf{R}},t)$$

$$\underline{\mathbf{R}} = x\underline{\mathbf{e}}_x + z\underline{\mathbf{e}}_z$$

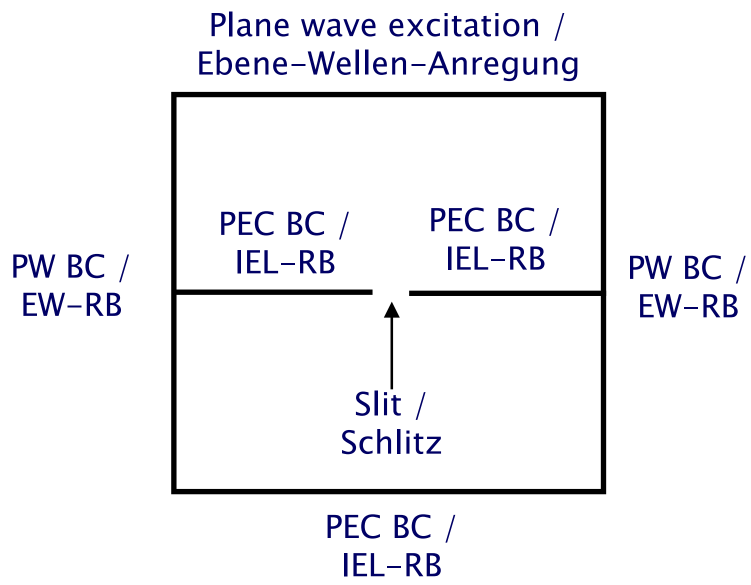
## Two-dimensional staggered grid system in the 2-D TM case / Zweidimensionales versetztes Gittersystem im 2D-TM-Fall



# Implementation of Boundary Conditions / Implementierung von Randbedingungen

**Boundary condition for a perfectly electrically conducting (PEC) material / Randbedingung für ein ideal elektrisch leitendes Material**

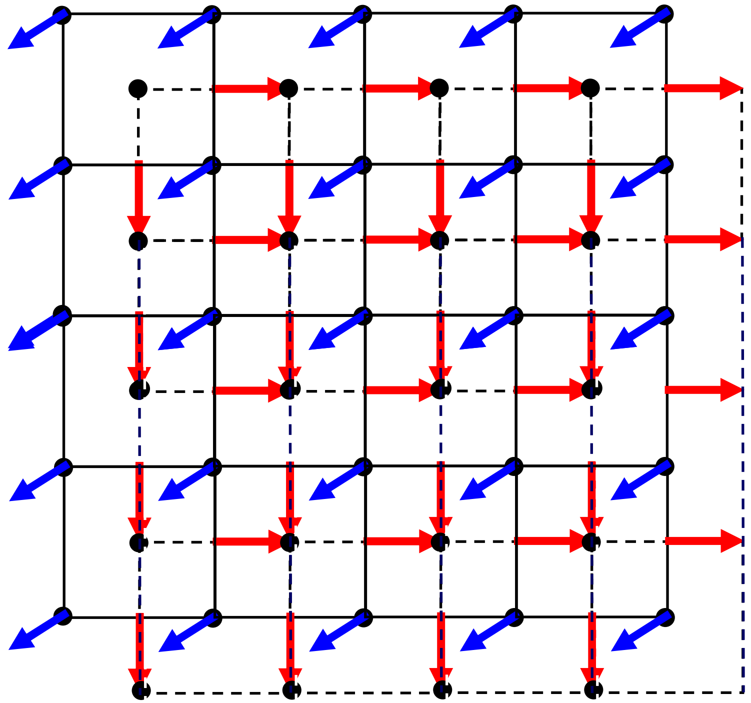
$$\left.\begin{aligned} E_y^{(\bullet,\bullet,n_t)} &= 0 \\ E_y^{(\bullet,\bullet,n_t)} &= 0 \end{aligned}\right\} \quad 1 \le n_t \le N_t$$

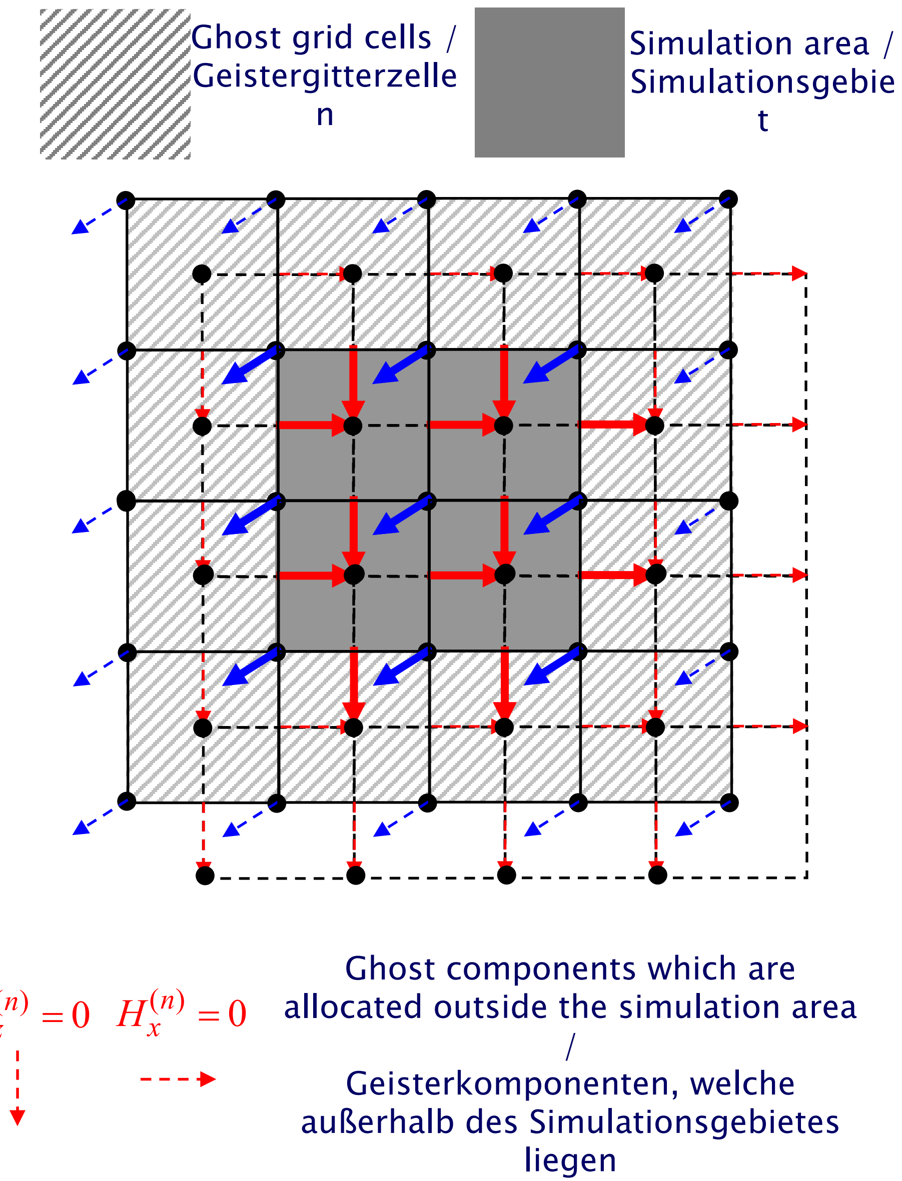



**Plane wave boundary condition for a vertical incident plane wave / Ebene-Wellen-Randbedingung für eine vertikal einfallende ebene Welle**

$$\left.\begin{aligned} E_y^{(2,n_z,n_t)} &= E_y^{(3,n_z,n_t)} \\ E_y^{(N_x-1,n_z,n_t)} &= E_y^{(N_x-2,n_z,n_t-2)} \end{aligned}\right\} \quad \begin{aligned} 1 \le n_z \le N_z \\ 1 \le n_t \le N_t \end{aligned}$$

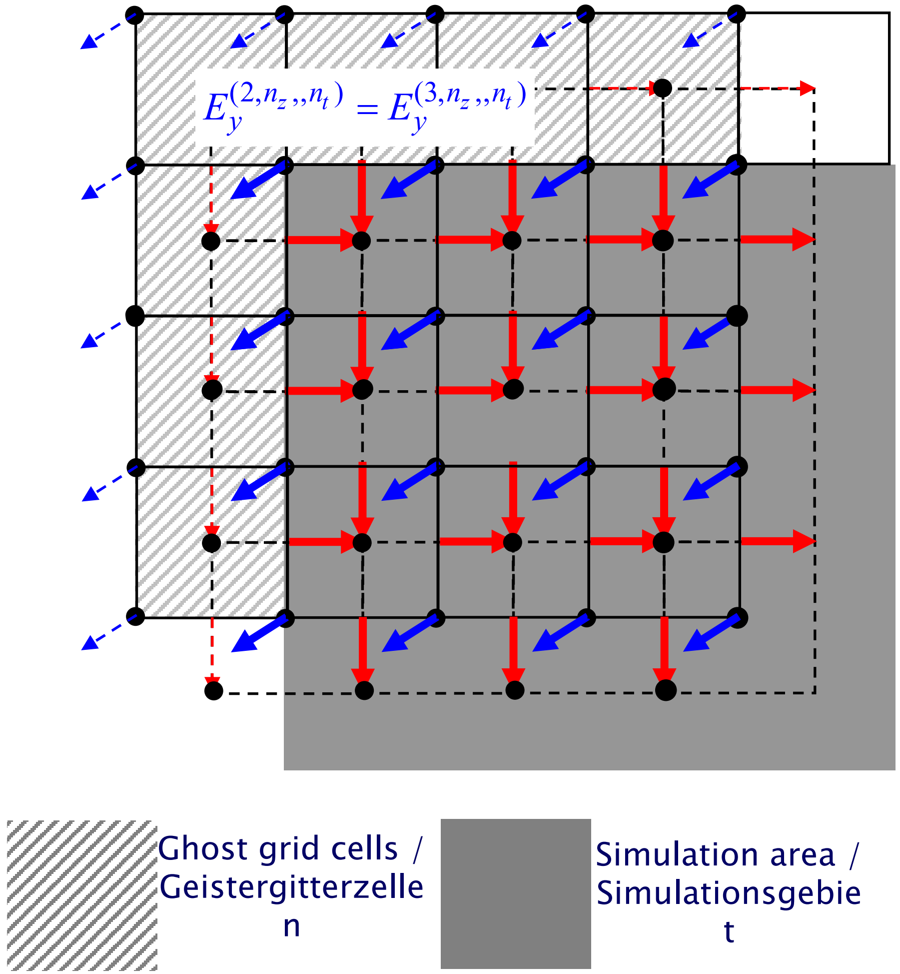
# 2-D EM Wave Propagation – 2-D FDTD – TM Case/ 2D EM Wellenausbreitung – 2D-FDTD – TM-Fall



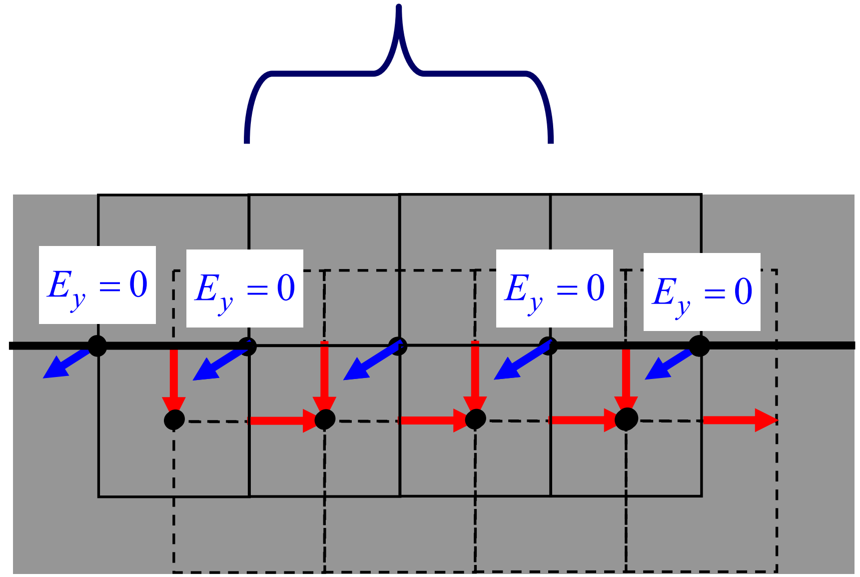


# 2-D EM Wave Propagation – 2-D FDTD – TM Case/ 2D EM Wellenausbreitung – 2D-FDTD – TM-Fall

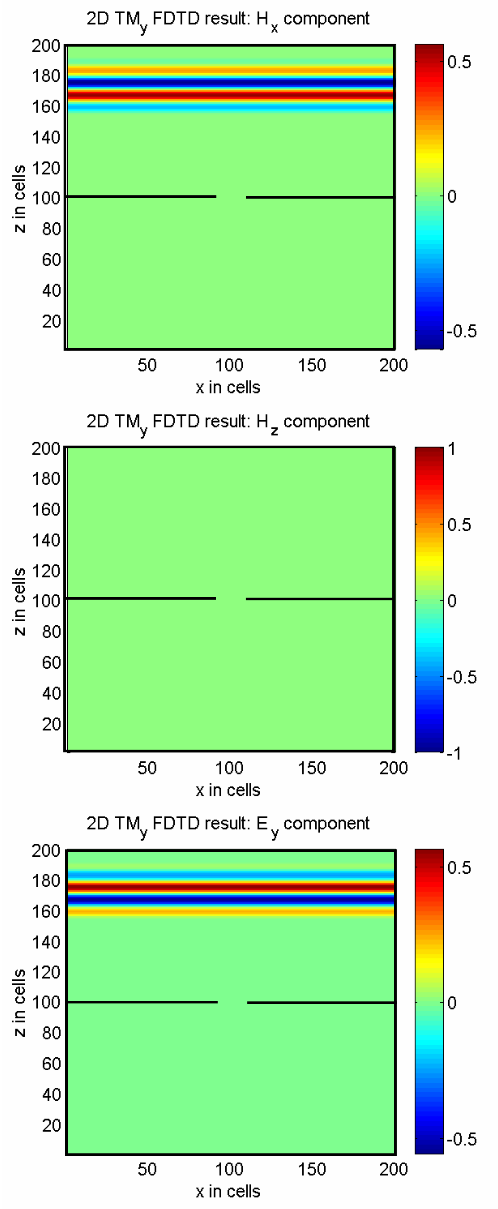
Plane wave excitation / Ebene-Wellen-Anregung

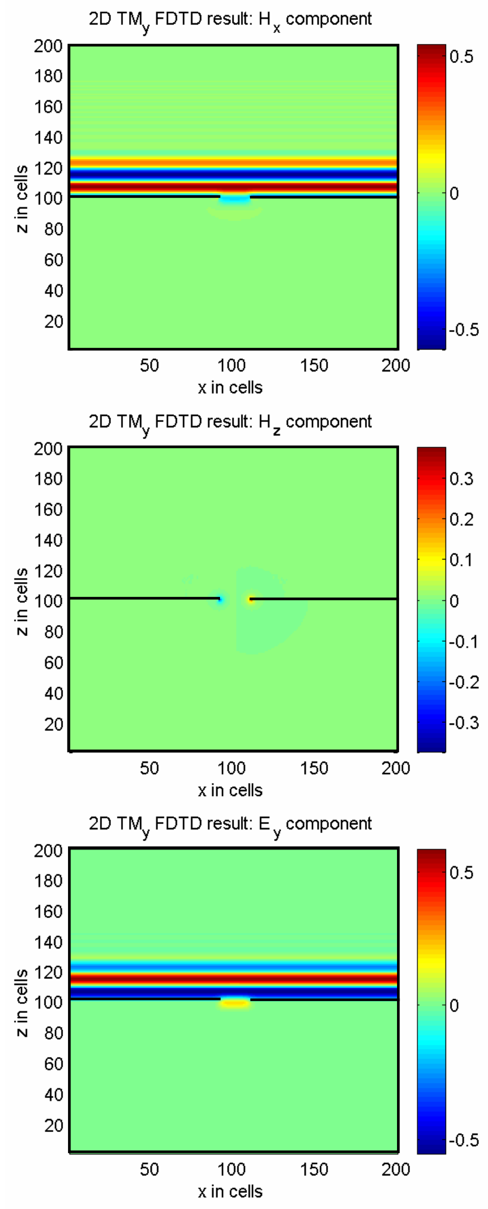


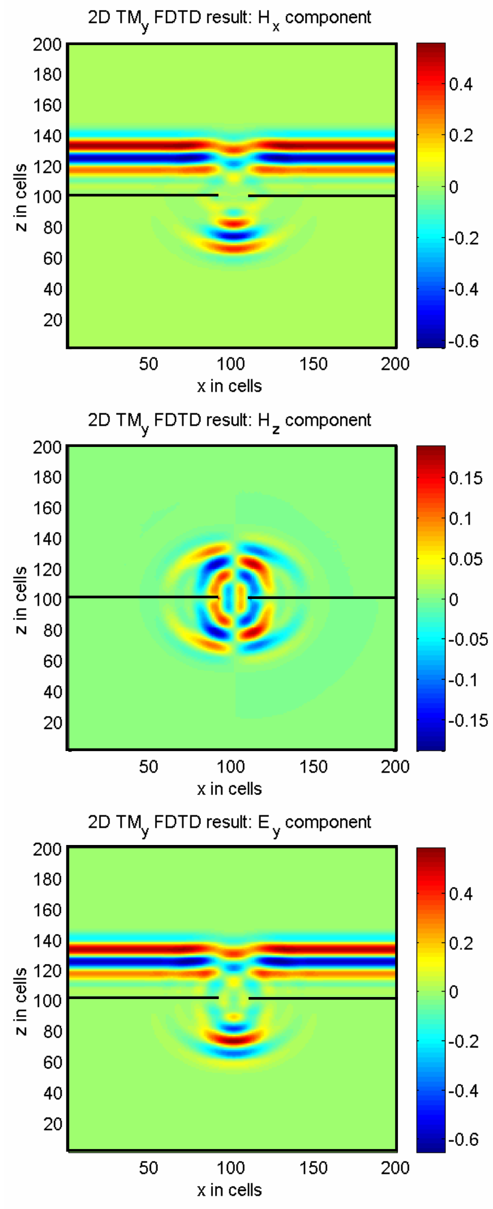

Slit / Schlitz



# 2-D TM FDTD – Diffraction on a Single Slit / 2D-TM-FDTD – Beugung an einem Spalt



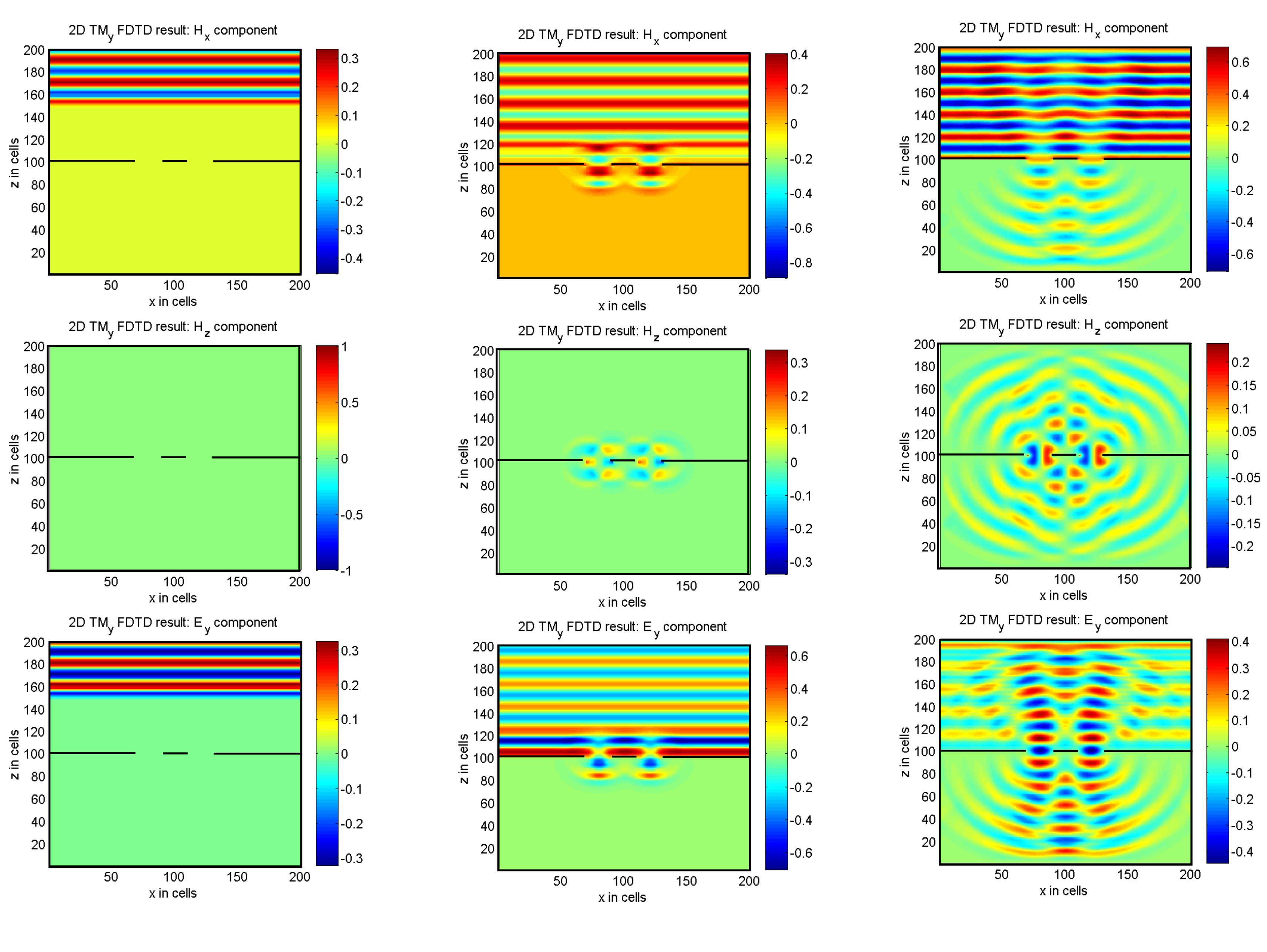
# 2–D TM FDTD – Diffraction on a Single Slit / 2D–TM–FDTD – Beugung am Spalt

Wave field movie of the $H_x$ field component / Wellenfeldfilm der $H_x$–Feldkomponente

Wave field movie of the $H_z$ field component / Wellenfeldfilm der $H_z$–Feldkomponente

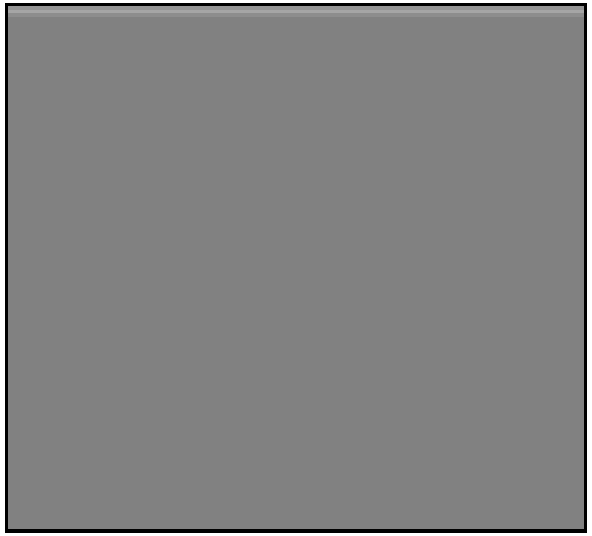
Wave field movie of the $E_y$ field component / Wellenfeldfilm der $E_y$–Feldkomponente

# 2-D TM FDTD – Diffraction on a Double Slit / 2D-TM-FDTD – Beugung am Doppelspalt

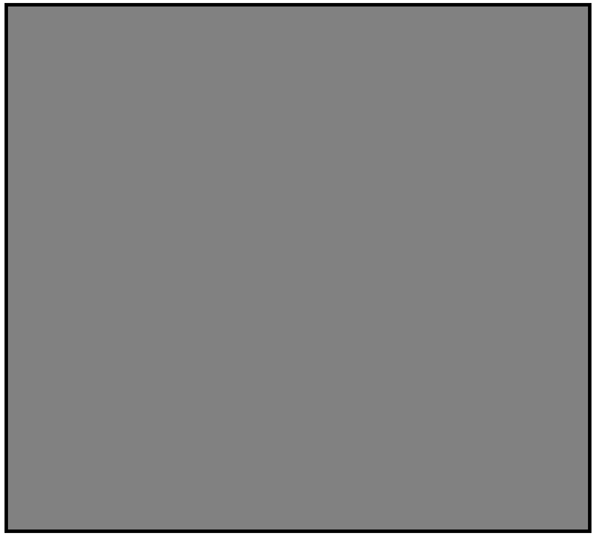


# 2–D TM FDTD – Diffraction on a Double Slit / 2D–TM–FDTD – Beugung am Doppelspalt

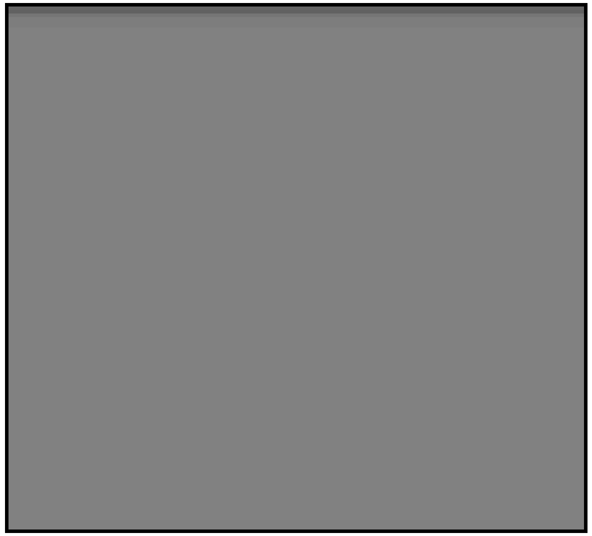
Wave field movie of the $H_x$ field component / Wellenfeldfilm der $H_x$–Feldkomponente



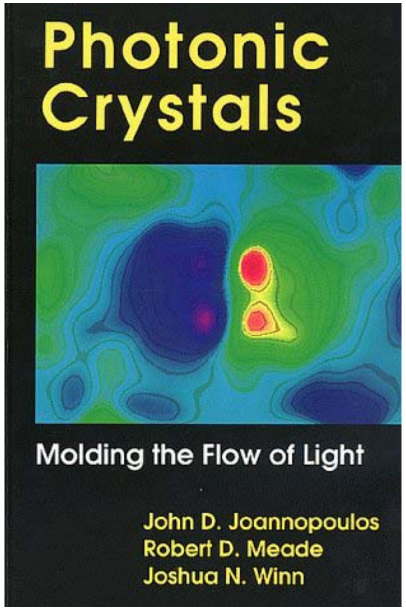
Wave field movie of the $H_z$ field component / Wellenfeldfilm der $H_z$–Feldkomponente



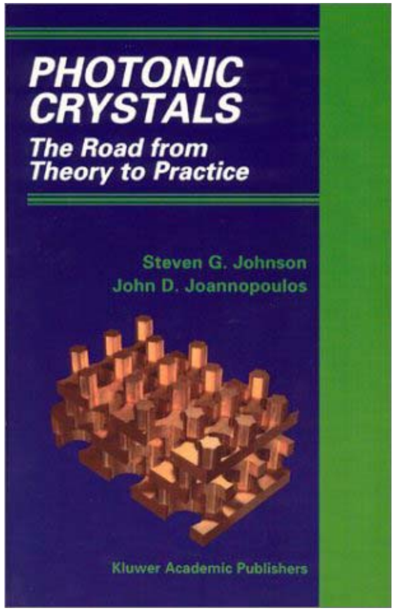
Wave field movie of the $E_y$ field component / Wellenfeldfilm der $E_y$–Feldkomponente



# Photonic Crystals / Photonische Kristalle




Joannopoulos, J. D., R. D. Meade, J. N. Winn: *Photonic Crystals – Molding the Flow of Light.* *Princeton University Press, Princeton, 1995.*




Johnson, S. G.: *Photonic Crystals: The Road from Theory to Practice*. Kluwer Academic Press, 2001.

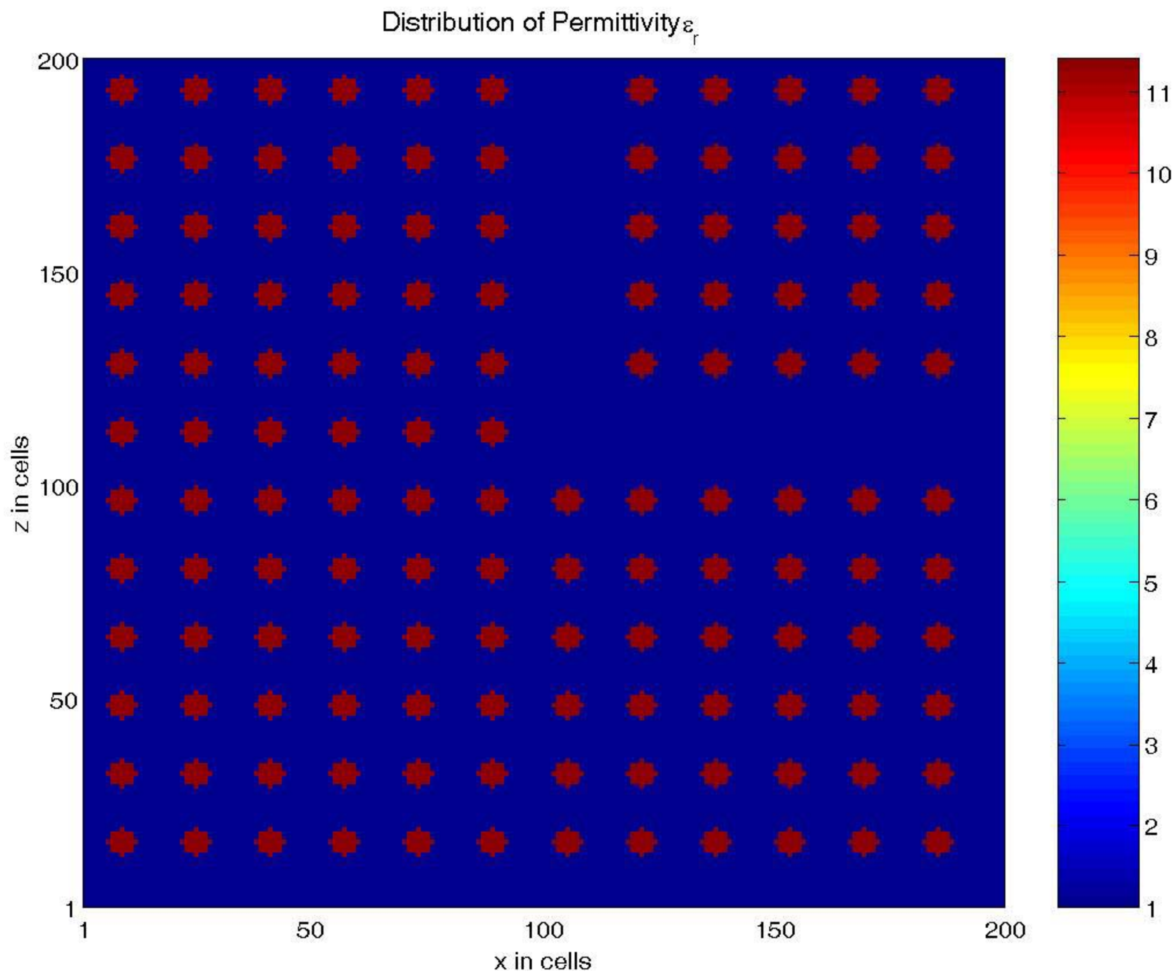
Links:

[Photonic Crystals Research at MIT]
[Homepage of Prof. Sajeev John, University of Toronto, Canada]

# 2-D TM FDTD – Photonic Crystals / 2D-TM-FDTD – Photonische Kristalle

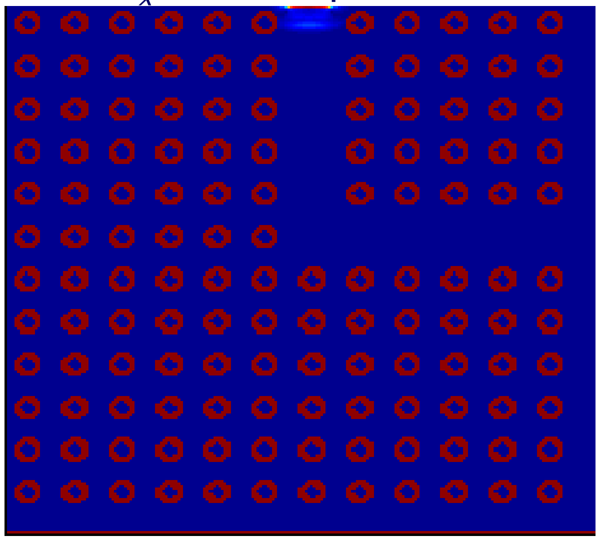
Relative permittivity of the background
Relative Permittivität des Hintergrundes $\varepsilon_r^{(b)} = 1$

Relative permittivity of the rods
Relative Permittivität der Stäbe $\varepsilon_r^{(r)} = 11.4$

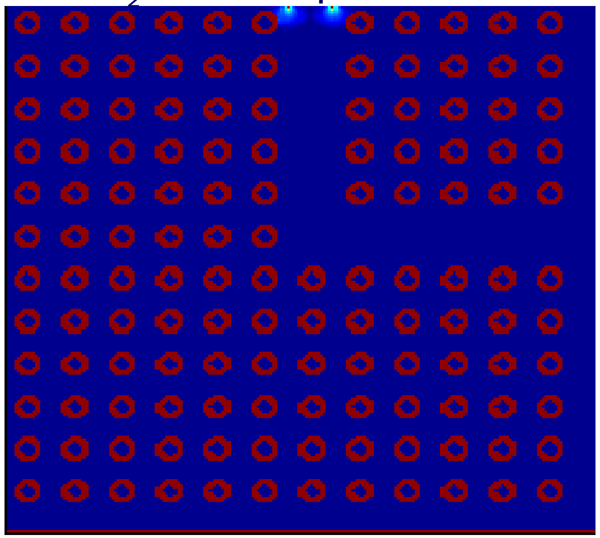


# 2-D TM FDTD – Photonic Crystals / 2D-TM-FDTD – Photonische Kristalle

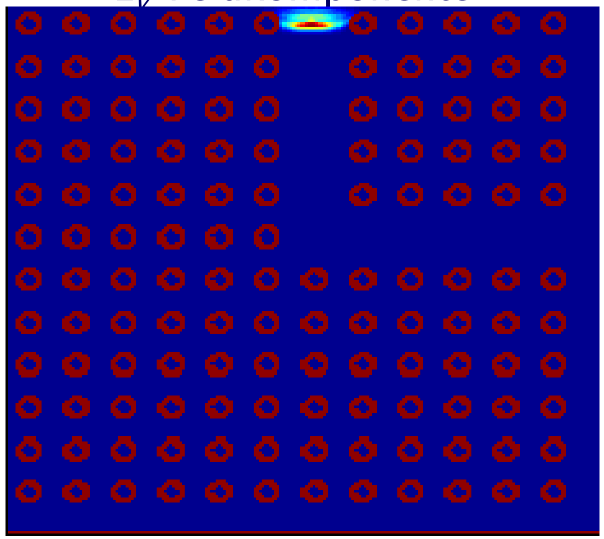
Wave field movie of the $H_x$ field component / Wellenfeldfilm der $H_x$-Feldkomponente



Wave field movie of the $H_z$ field component / Wellenfeldfilm der $H_z$-Feldkomponente



Wave field movie of the $E_y$ field component / Wellenfeldfilm der $E_y$-Feldkomponente



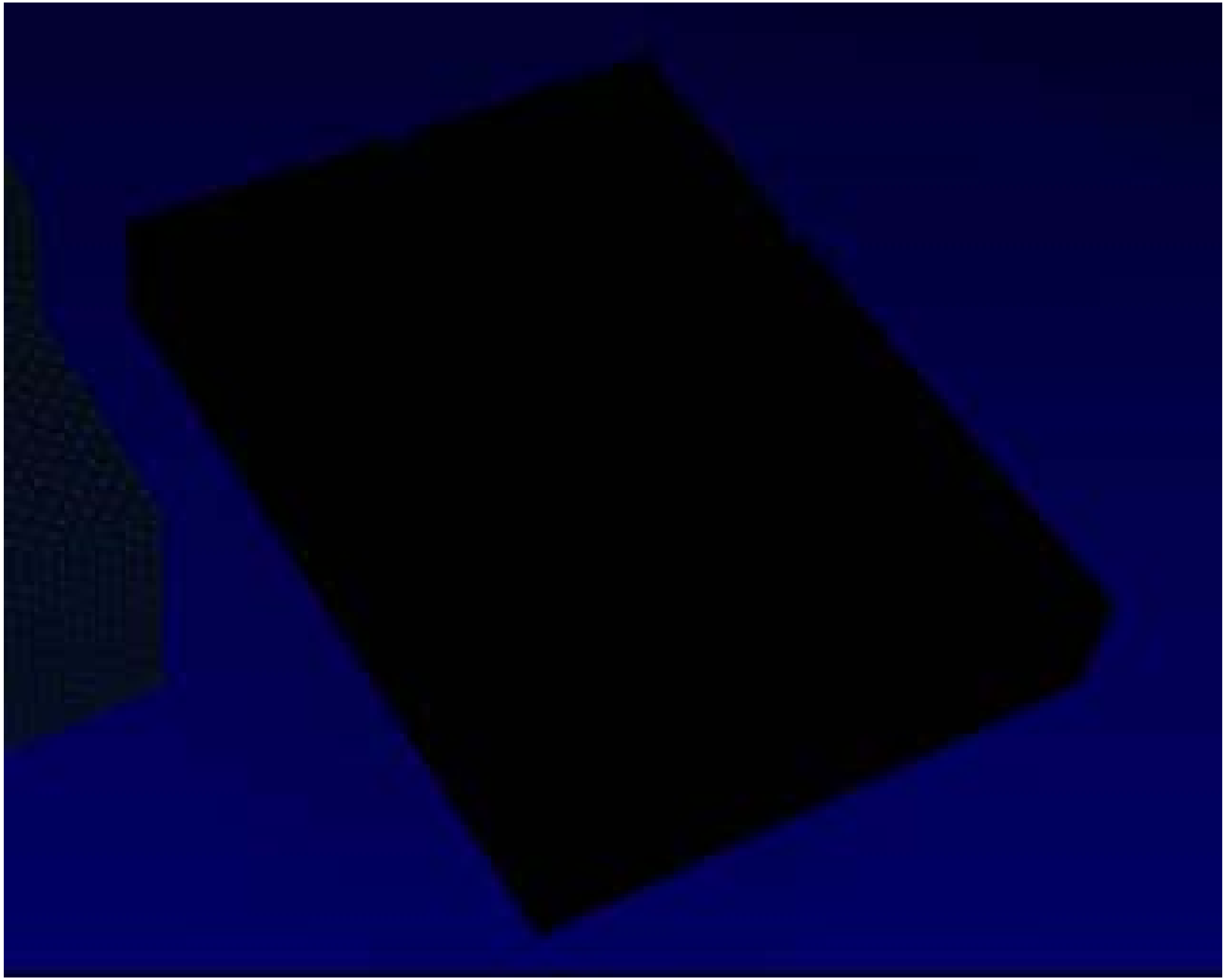
# 2-D TM FDTD – Photonic Crystals / 2D-TM-FDTD – Photonische Kristalle

Wave field movie of the $H_x$ field component / Wellenfeldfilm der $H_x$-Feldkomponente

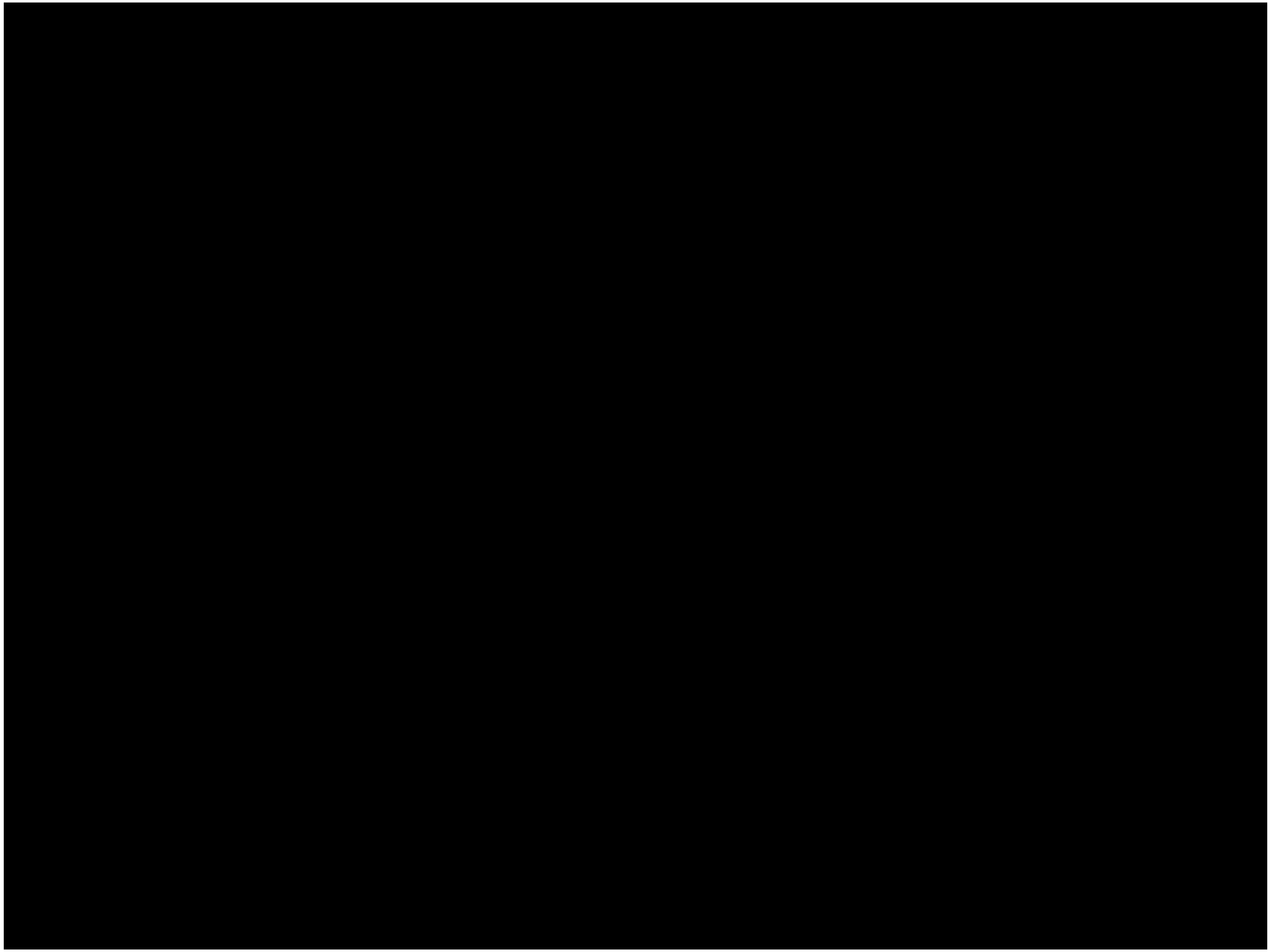
Wave field movie of the $H_z$ field component / Wellenfeldfilm der $H_z$-Feldkomponente

Wave field movie of the $E_y$ field component / Wellenfeldfilm der $E_y$-Feldkomponente

# 2-D TM FDTD – Photonic Crystals / 2D-TM-FDTD – Photonische Kristalle



# 2-D TM FDTD – Photonic Crystals /
# 2D-TM-FDTD – Photonische Kristalle



# FDTD and FIT / FDTD und FIT

FDTD : Finite Difference Time Domain / Finite Differenzen im Zeitbereich
FIT : Finite Integration Technique / Finite Integrationstechnik

## FDTD

Maxwell's equations in differential form / Maxwellsche Gleichungen in Differentialform

$$\frac{\partial}{\partial t}\underline{\mathbf{B}}(\underline{\mathbf{R}},t) = -\nabla\times\underline{\mathbf{E}}(\underline{\mathbf{R}},t) - \underline{\mathbf{J}}_{\mathrm{m}}(\underline{\mathbf{R}},t)$$

$$\frac{\partial}{\partial t}\underline{\mathbf{D}}(\underline{\mathbf{R}},t) = \nabla\times\underline{\mathbf{H}}(\underline{\mathbf{R}},t) - \underline{\mathbf{J}}_{\mathrm{e}}(\underline{\mathbf{R}},t)$$

$$\nabla\bullet\underline{\mathbf{D}}(\underline{\mathbf{R}},t) = \rho_{\mathrm{e}}(\underline{\mathbf{R}},t)$$

$$\nabla\bullet\underline{\mathbf{B}}(\underline{\mathbf{R}},t) = \rho_{\mathrm{m}}(\underline{\mathbf{R}},t)$$

⇨ FD approximation of spatial and temporal derivatives / FD-Approximation von räumlichen und zeitlichen Ableitungen

Central difference approximation / Zentrale Differenzen Approximation

$$\left.\frac{\partial}{\partial z}f(z,t)\right|_{z=z_0} \approx \frac{f\left(z_0+\frac{\Delta z}{2},t\right)-f\left(z_0-\frac{\Delta z}{2},t\right)}{\Delta z}$$

## FIT

Maxwell's equations in integral form / Maxwellsche Gleichungen in Integralform

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint_S \underline{\mathbf{B}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = -\oint_{C=\partial S}\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dR}} - \iint_S \underline{\mathbf{J}}_{\mathrm{m}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint_S \underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = \oint_{C=\partial S}\underline{\mathbf{H}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dR}} - \iint_S \underline{\mathbf{J}}_{\mathrm{e}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}}$$

$$\oiint_{S=\partial V}\underline{\mathbf{D}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = \iiint_V \rho_{\mathrm{e}}(\underline{\mathbf{R}},t)\,dV$$

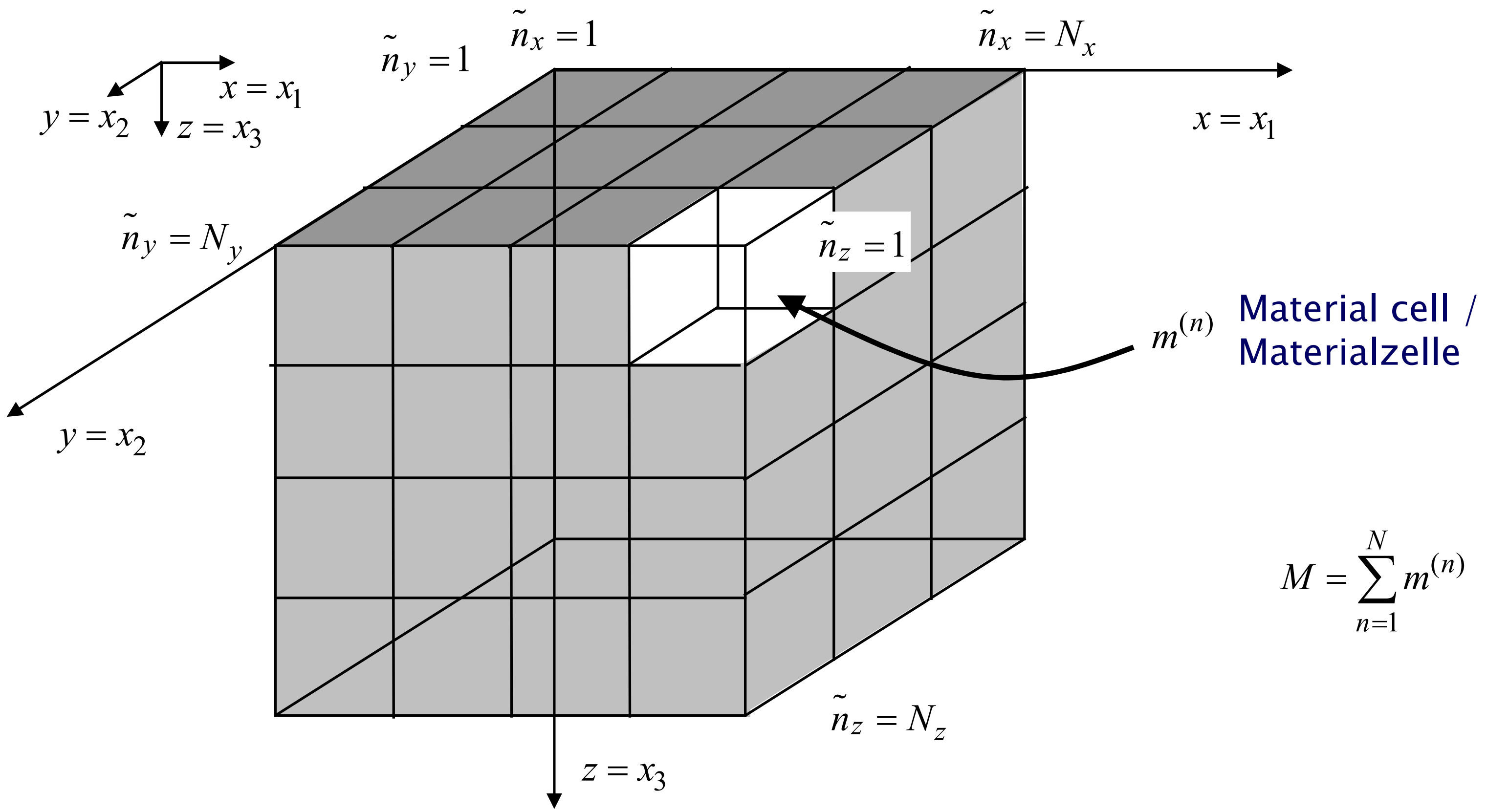
$$\oiint_{S=\partial V}\underline{\mathbf{B}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = \iiint_V \rho_{\mathrm{m}}(\underline{\mathbf{R}},t)\,dV$$

⇨ FIT approximation of spatial and temporal integrals / FIT-Approximation von räumlichen und zeitlichen Integralen

Mid point rule approximation of a 1-D integral / Mittelpunktsregel-Approximation eines 1D-Integrals

$$\int_{z=z_0}^{z_0+\Delta z} f(z,t)\,\mathrm{d}z \approx f\left(z_0+\frac{\Delta z}{2},t\right)\Delta z$$

# Definition of Material Cells / Definition der Materialzellen




$$M = \sum_{n=1}^{N} m^{(n)}$$

$$n = 1 + M_x\left(n_x - 1\right) + M_y\left(n_y - 1\right) + M_z\left(n_z - 1\right)$$

$$n = 1, 2, \ldots, N = N_x N_y N_z$$

$$M_x = 1$$

$$M_y = N_x$$

$$M_z = N_x N_y$$

$$\underline{\underline{\boldsymbol{\varepsilon}}}(\underline{\mathbf{R}}) \rightarrow \underline{\underline{\boldsymbol{\varepsilon}}}^{(n)} \in m^{(n)} \quad n \in \mathbb{R}^N$$

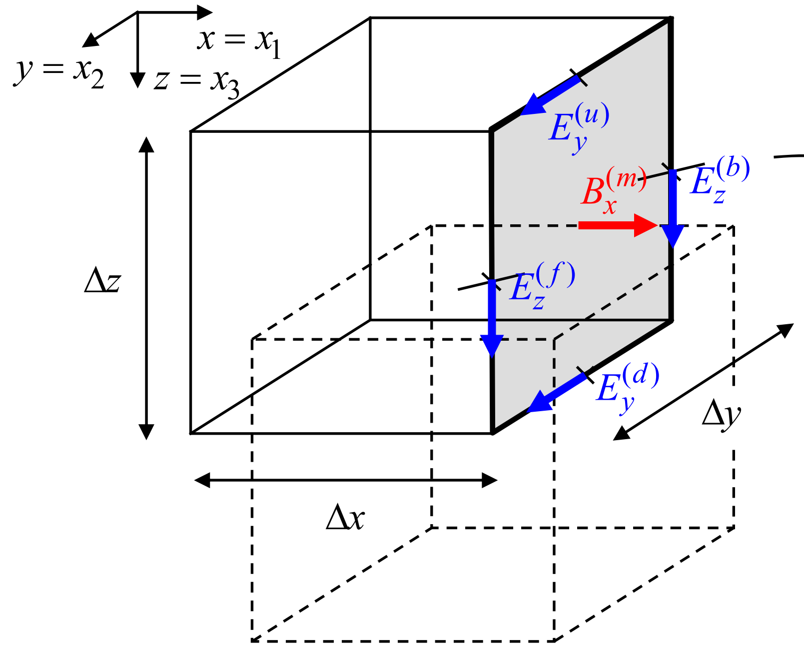
$$\underline{\underline{\boldsymbol{\nu}}}(\underline{\mathbf{R}}) \rightarrow \underline{\underline{\boldsymbol{\nu}}}^{(n)} \in m^{(n)} \quad n \in \mathbb{R}^N$$

# 3-D FIT – Derivation of the Discrete Grid Equations / 3D-FIT – Ableitung der diskreten Gittergleichungen

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint_S \underline{\mathbf{B}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = -\oint_{C=\partial S}\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dR}} - \iint_S \underline{\mathbf{J}}_\mathrm{m}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}}$$

$$\underline{\mathbf{dS}} = \underline{\mathbf{n}}\,\mathrm{d}S = \underline{\mathbf{e}}_x\,\mathrm{d}S$$

$$\underline{\mathbf{dR}} = \underline{\mathbf{s}}\,\mathrm{d}R$$



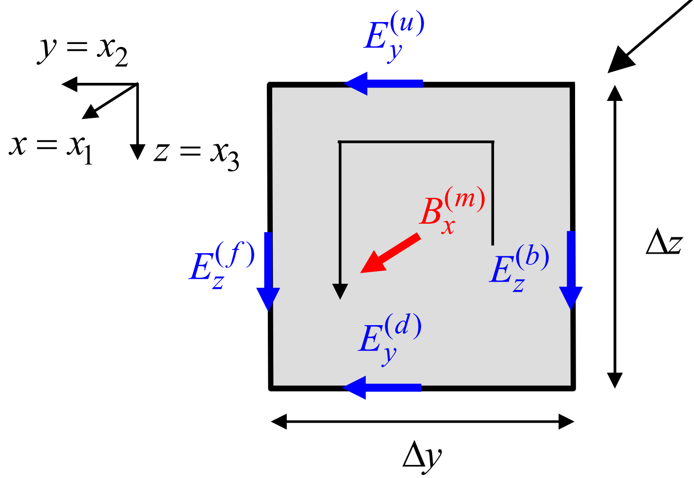

$$\iint_S \underline{\mathbf{n}}\bullet\underline{\mathbf{B}}(\underline{\mathbf{R}},t)\,dS = \iint_S \underline{\mathbf{e}}_x\bullet\underline{\mathbf{B}}(\underline{\mathbf{R}},t)\,\mathrm{d}S$$

$$= \iint_S B_x(\underline{\mathbf{R}},t)\,\mathrm{d}S$$

$$= B_x^{(m)}(t)\underbrace{\iint_S \mathrm{d}S}_{=\Delta y\Delta z} + \mathrm{O}\left[(\Delta y)^3\Delta z + \Delta y(\Delta z)^3\right]$$

$$= B_x^{(m)}(t)\Delta y\Delta z + \mathrm{O}\left[(\Delta y)^3\Delta z + \Delta y(\Delta z)^3\right]$$

$I_{B_x^{(m)}}$-Integration cell / $I_{B_x^{(m)}}$ -Integrationszelle




Field component in the middle / Feldkomponente in der Mitte

Approximation error / Approximationsfehler

$$\iint_S f(\underline{\mathbf{R}},t)\,\mathrm{d}S = f^{(m)}(t)\underbrace{\iint_S \mathrm{d}y\,\mathrm{d}z}_{=\Delta y\Delta z} + \mathrm{O}\left[(\Delta y)^3\Delta z + \Delta y(\Delta z)^3\right]$$

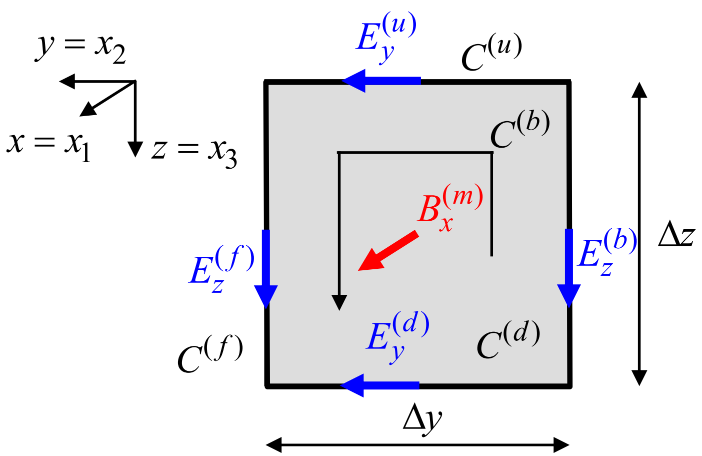
$$= f^{(m)}(t)\Delta y\Delta z + \mathrm{O}\left[(\Delta y)^3\Delta z + \Delta y(\Delta z)^3\right]$$

# 3-D FIT – Derivation of the Discrete Grid Equations / 3D-FIT – Ableitung der diskreten Gittergleichungen

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint_S \underline{\mathbf{B}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = -\oint_{C=\partial S}\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dR}} - \iint_S \underline{\mathbf{J}}_\mathrm{m}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}}$$

$$\oint_{C=\partial S}\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dR}} = ?$$

$I_{B_x^{(m)}}$ integration cell / $I_{B_x^{(m)}}$ -Integrationszelle




$$\underline{\mathbf{dS}} = \underline{\mathbf{n}}\,\mathrm{d}S = \underline{\mathbf{e}}_x\,\mathrm{d}y\,\mathrm{d}z$$

$$\underline{\mathbf{dR}}_y = \underline{\mathbf{s}}\,\mathrm{d}R = \underline{\mathbf{e}}_y\,\mathrm{d}y$$

$$\underline{\mathbf{dR}}_z = \underline{\mathbf{s}}\,\mathrm{d}R = \underline{\mathbf{e}}_z\,\mathrm{d}z$$

$$\begin{aligned}\oint_{C=\partial S}\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dR}} &= \int_{C^{(u)}}\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dR}} + \int_{C^{(f)}}\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dR}} \\ &\quad + \int_{C^{(d)}}\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dR}} + \int_{C^{(b)}}\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dR}} \\ &= \int_{C^{(u)}}\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{e}}_y\mathrm{d}y + \int_{C^{(f)}}\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{e}}_z\mathrm{d}z \\ &\quad - \int_{C^{(d)}}\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{e}}_y\mathrm{d}y - \int_{C^{(b)}}\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{e}}_z\mathrm{d}z \\ &= \int_{C^{(u)}}E_y(\underline{\mathbf{R}},t)\,\mathrm{d}y + \int_{C^{(f)}}E_z(\underline{\mathbf{R}},t)\,\mathrm{d}z \\ &\quad - \int_{C^{(d)}}E_y(\underline{\mathbf{R}},t)\,\mathrm{d}y - \int_{C^{(b)}}E_z(\underline{\mathbf{R}},t)\,\mathrm{d}z\end{aligned}$$

# 3-D FIT – Derivation of the Discrete Grid Equations / 3D-FIT – Ableitung der diskreten Gittergleichungen

$$\oint_{C=\partial S} \underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet \underline{\mathbf{dR}} = \int_{C^{(u)}} E_y(\underline{\mathbf{R}},t)\,\mathrm{d}y + \int_{C^{(f)}} E_z(\underline{\mathbf{R}},t)\,\mathrm{d}z - \int_{C^{(d)}} E_y(\underline{\mathbf{R}},t)\,\mathrm{d}y - \int_{C^{(b)}} E_z(\underline{\mathbf{R}},t)\,\mathrm{d}z$$

$$\int_{C^{(u)}} E_y(\underline{\mathbf{R}},t)\,\mathrm{d}y = E_y^{(u)}(t)\underbrace{\int_{C^{(u)}} \mathrm{d}y}_{=\Delta y} + \bigcirc\left[(\Delta y)^3\right] = E_y^{(u)}(t)\Delta y + \bigcirc\left[(\Delta y)^3\right]$$

$$\int_{C^{(f)}} E_z(\underline{\mathbf{R}},t)\,\mathrm{d}z = E_z^{(f)}(t)\underbrace{\int_{C^{(f)}} \mathrm{d}z}_{=\Delta z} + \bigcirc\left[(\Delta z)^3\right] = E_z^{(f)}(t)\Delta z + \bigcirc\left[(\Delta z)^3\right]$$

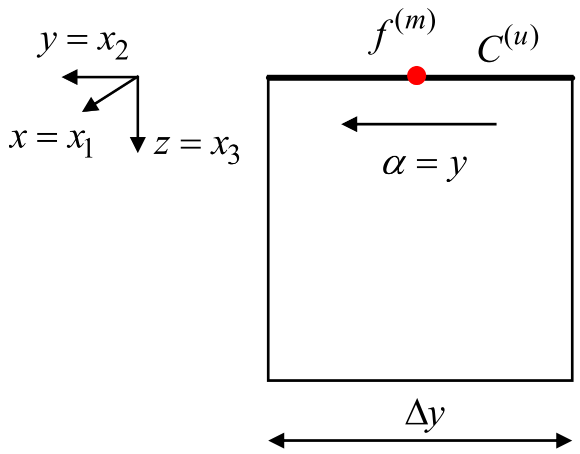
$$\int_{C^{(d)}} E_y(\underline{\mathbf{R}},t)\,\mathrm{d}y = E_y^{(d)}(t)\underbrace{\int_{C^{(d)}} \mathrm{d}y}_{=\Delta y} + \bigcirc\left[(\Delta y)^3\right] = E_y^{(d)}(t)\Delta y + \bigcirc\left[(\Delta y)^3\right]$$

$$\int_{C^{(b)}} E_z(\underline{\mathbf{R}},t)\,\mathrm{d}z = E_z^{(b)}(t)\underbrace{\int_{C^{(b)}} \mathrm{d}z}_{=\Delta z} + \bigcirc\left[(\Delta z)^3\right] = E_z^{(b)}(t)\Delta z + \bigcirc\left[(\Delta z)^3\right]$$

Field component in the middle / Feldkomponente in der Mitte

Approximation error / Approximationsfehler

$$\int_{C^{(u)}} f(\underline{\mathbf{R}},t)\,\mathrm{d}R = f^{(m)}(t)\underbrace{\int_{C^{(u)}} \mathrm{d}y}_{=\Delta y} + \bigcirc\left[(\Delta y)^3\right] = f^{(m)}(t)\Delta y + \bigcirc\left[(\Delta y)^3\right]$$



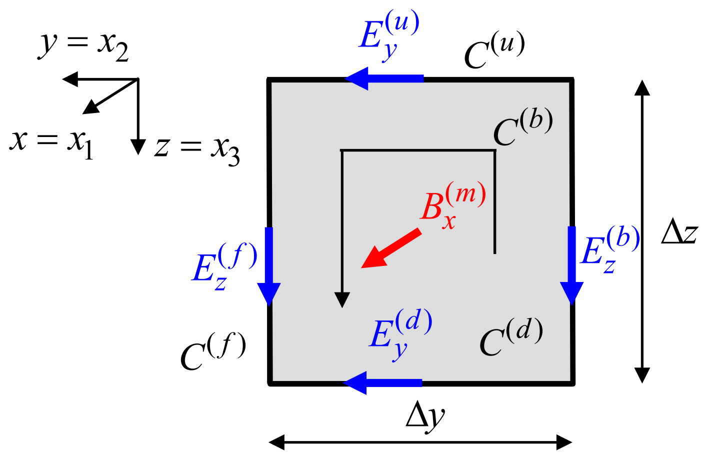
# 3-D FIT – Derivation of the Discrete Grid Equations / 3D-FIT – Ableitung der diskreten Gittergleichungen

$$\oint_{C=\partial S} \underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet \underline{\mathbf{dR}} = \int_{C^{(u)}} E_y(\underline{\mathbf{R}},t)\,\mathrm{d}y + \int_{C^{(f)}} E_z(\underline{\mathbf{R}},t)\,\mathrm{d}z - \int_{C^{(d)}} E_y(\underline{\mathbf{R}},t)\,\mathrm{d}y - \int_{C^{(b)}} E_z(\underline{\mathbf{R}},t)\,\mathrm{d}z$$

$$= E_y^{(u)}(t)\underbrace{\int_{C^{(u)}} \mathrm{d}y}_{=\Delta y} + E_z^{(f)}(t)\underbrace{\int_{C^{(f)}} \mathrm{d}z}_{=\Delta z} - E_y^{(d)}(t)\underbrace{\int_{C^{(d)}} \mathrm{d}y}_{=\Delta y} - E_z^{(b)}(t)\underbrace{\int_{C^{(b)}} \mathrm{d}z}_{=\Delta z}$$

$$+\bigcirc\left[(\Delta y)^3\right] + \bigcirc\left[(\Delta z)^3\right]$$

$I_{B_x^{(m)}}$ integration cell / $I_{B_x^{(m)}}$ –Integrationszelle




$$\oint_{C=\partial S} \underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet \underline{\mathbf{dR}} = E_y^{(u)}(t)\Delta y + E_z^{(f)}(t)\Delta z - E_y^{(d)}(t)\Delta y - E_z^{(b)}(t)\Delta z + \bigcirc\left[(\Delta y)^3\right] + \bigcirc\left[(\Delta z)^3\right]$$

# 3-D FIT – Derivation of the Discrete Grid Equations / 3D-FIT – Ableitung der diskreten Gittergleichungen

$I_{B_x^{(m)}}$-integration cell / $I_{B_x^{(m)}}$-Integrationszelle

$$\oint_{C=\partial S} \underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet \underline{\mathbf{dR}} = E_y^{(u)}(t)\Delta y + E_z^{(f)}(t)\Delta z - E_y^{(d)}(t)\Delta y - E_z^{(b)}(t)\Delta z + \mathrm{O}\left[(\Delta y)^3\right] + \mathrm{O}\left[(\Delta z)^3\right]$$

$I_{B_x^{(m)}}$-integration cell / $I_{B_x^{(m)}}$-Integrationszelle

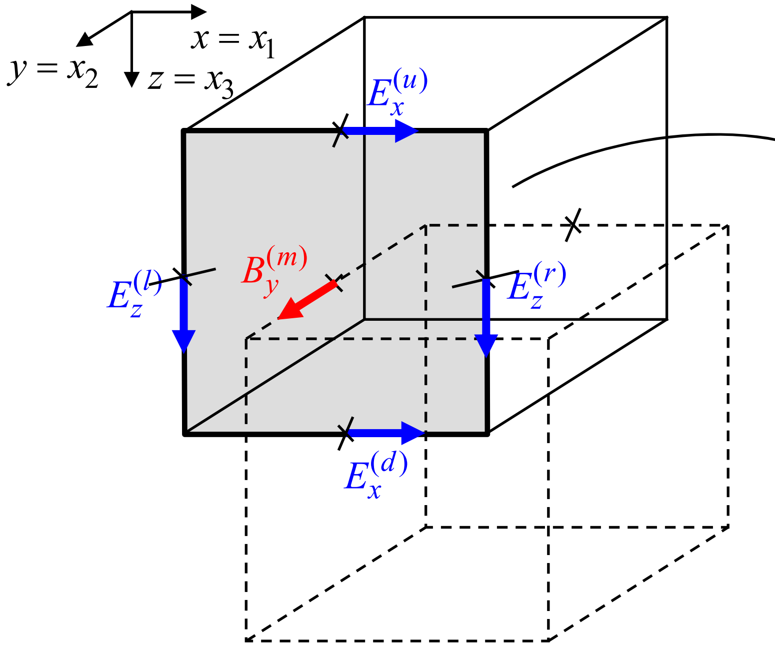
$$\begin{aligned}\iint_S \underline{\mathbf{n}}\bullet \underline{\mathbf{J}}_{\mathrm{m}}(\underline{\mathbf{R}},t)\, dS &= \iint_S \underline{\mathbf{e}}_x \bullet \underline{\mathbf{J}}_{\mathrm{m}}(\underline{\mathbf{R}},t)\,\mathrm{d}S \\ &= \iint_S J_{\mathrm{m}x}(\underline{\mathbf{R}},t)\,\mathrm{d}S \\ &= J_{\mathrm{m}x}^{(m)}(t)\underbrace{\iint_S \mathrm{d}S}_{=\Delta y\Delta z} + \mathrm{O}\left[(\Delta y)^3\Delta z + \Delta y(\Delta z)^3\right] \\ &= J_{\mathrm{m}x}^{(m)}(t)\Delta y\Delta z + \mathrm{O}\left[(\Delta y)^3\Delta z + \Delta y(\Delta z)^3\right]\end{aligned}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint_S \underline{\mathbf{B}}(\underline{\mathbf{R}},t)\bullet \underline{\mathbf{dS}} = -\oint_{C=\partial S} \underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet \underline{\mathbf{dR}} - \iint_S \underline{\mathbf{J}}_{\mathrm{m}}(\underline{\mathbf{R}},t)\bullet \underline{\mathbf{dS}}$$

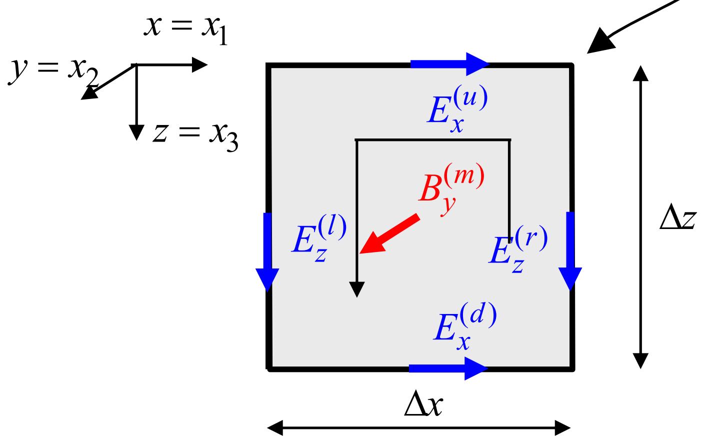
$I_{B_x^{(m)}}$-integration cell / $I_{B_x^{(m)}}$-Integrationszelle

$$\frac{\mathrm{d}}{\mathrm{d}t} B_x^{(m)}(t)\Delta y\Delta z = -\left[E_y^{(u)}(t)\Delta y + E_z^{(f)}(t)\Delta z - E_y^{(d)}(t)\Delta y - E_z^{(b)}(t)\Delta z\right] - J_{\mathrm{m}x}^{(m)}(t)\Delta y\Delta z$$

# 3-D FIT – Derivation of the Discrete Grid Equations / 3D-FIT – Ableitung der diskreten Gittergleichungen




$I_{B_y^{(m)}}$ integration cell / $I_{B_y^{(m)}}$ –Integrationszelle




$$\frac{\mathrm{d}}{\mathrm{d}t}\iint_S \underline{\mathbf{B}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = -\oint_{C=\partial S}\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dR}} - \iint_S \underline{\mathbf{J}}_{\mathrm{m}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}}$$

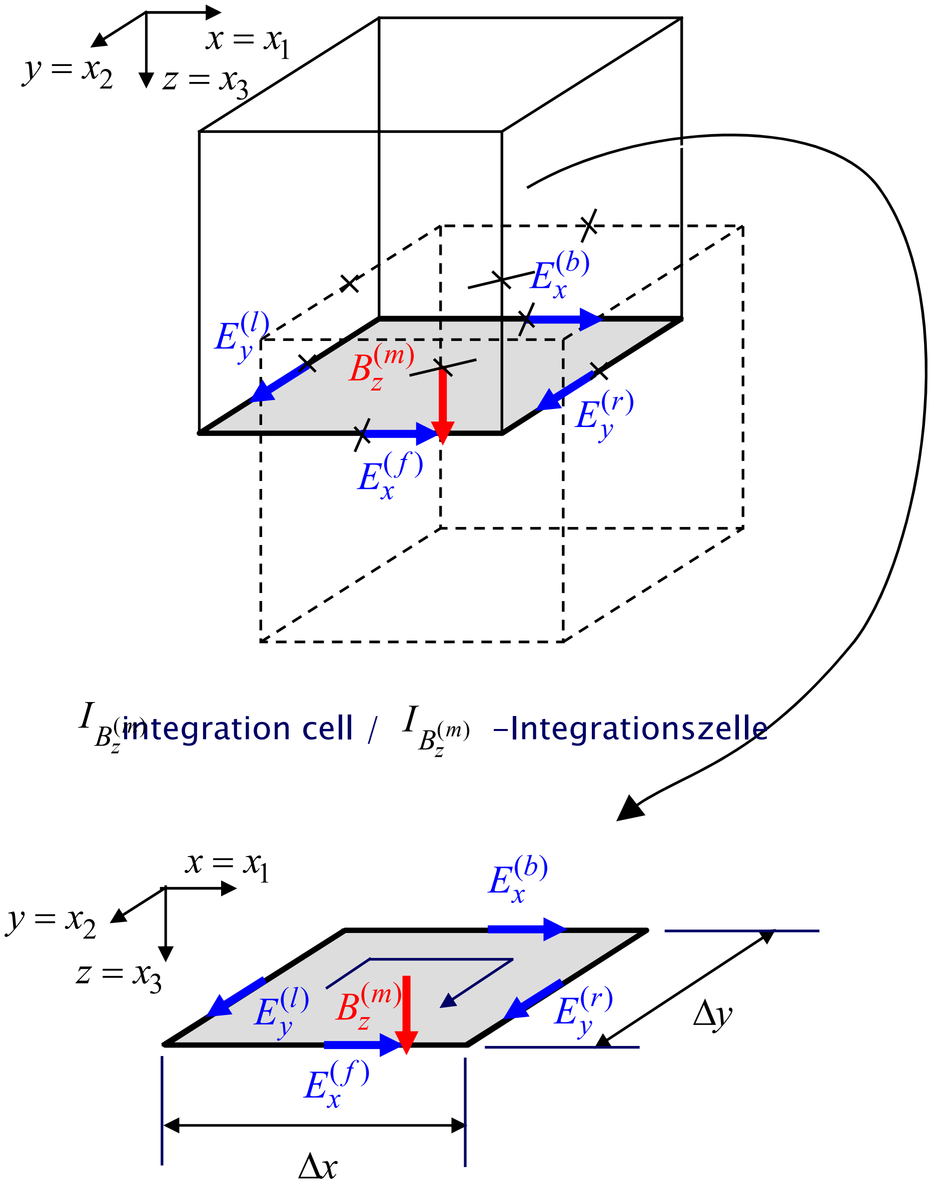
$I_{B_y^{(m)}}$ integration cell / $I_{B_y^{(m)}}$ –Integrationszelle

$$\frac{\mathrm{d}}{\mathrm{d}t}B_y^{(m)}(t)\Delta y\Delta z$$

$$= -\left[-E_x^{(u)}(t)\Delta x + E_z^{(l)}(t)\Delta z + E_x^{(d)}(t)\Delta x - E_z^{(r)}(t)\Delta z\right]$$

$$-J_{\mathrm{my}}^{(m)}(t)\Delta y\Delta z$$

# 3-D FIT – Derivation of the Discrete Grid Equations / 3D-FIT – Ableitung der diskreten Gittergleichungen




$$\frac{\mathrm{d}}{\mathrm{d}t}\iint_S \underline{\mathbf{B}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = -\oint_{C=\partial S}\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dR}} - \iint_S \underline{\mathbf{J}}_{\mathrm{m}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}}$$

$I_{B_z^{(m)}}$ integration cell / $I_{B_z^{(m)}}$ -Integrationszelle

$$\frac{\mathrm{d}}{\mathrm{d}t}B_z^{(m)}(t)\Delta x\Delta y$$

$$= -\left[E_x^{(b)}(t)\Delta x + E_y^{(r)}(t)\Delta y - E_x^{(f)}(t)\Delta x - E_y^{(l)}(t)\Delta y\right]$$

$$- J_{\mathrm{mz}}^{(m)}(t)\Delta x\Delta y$$

# 3-D FIT – Derivation of the Discrete Grid Equations / 3D-FIT – Ableitung der diskreten Gittergleichungen

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint_S \underline{\mathbf{B}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}} = -\oint_{C=\partial S}\underline{\mathbf{E}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dR}} - \iint_S \underline{\mathbf{J}}_{\mathrm{m}}(\underline{\mathbf{R}},t)\bullet\underline{\mathbf{dS}}$$

$$\Downarrow$$

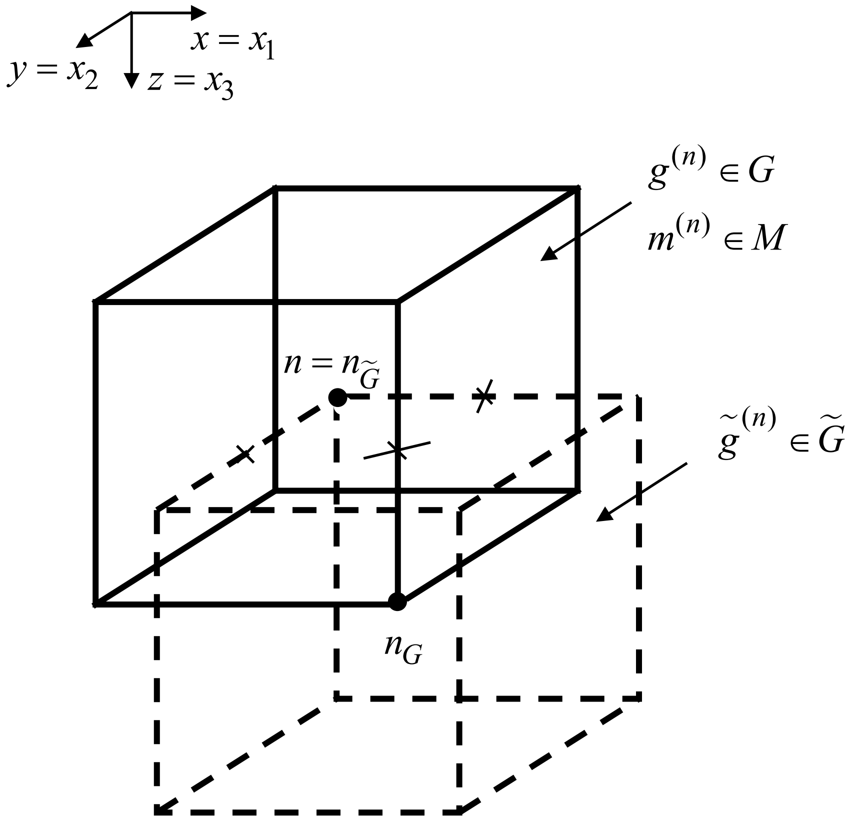
$$\frac{\mathrm{d}}{\mathrm{d}t}B_x^{(m)}(t)\Delta y\Delta z = -\left[E_y^{(u)}(t)\Delta y + E_z^{(f)}(t)\Delta z - E_y^{(d)}(t)\Delta y - E_z^{(b)}(t)\Delta z\right] - J_{\mathrm{m}x}^{(m)}(t)\Delta y\Delta z$$

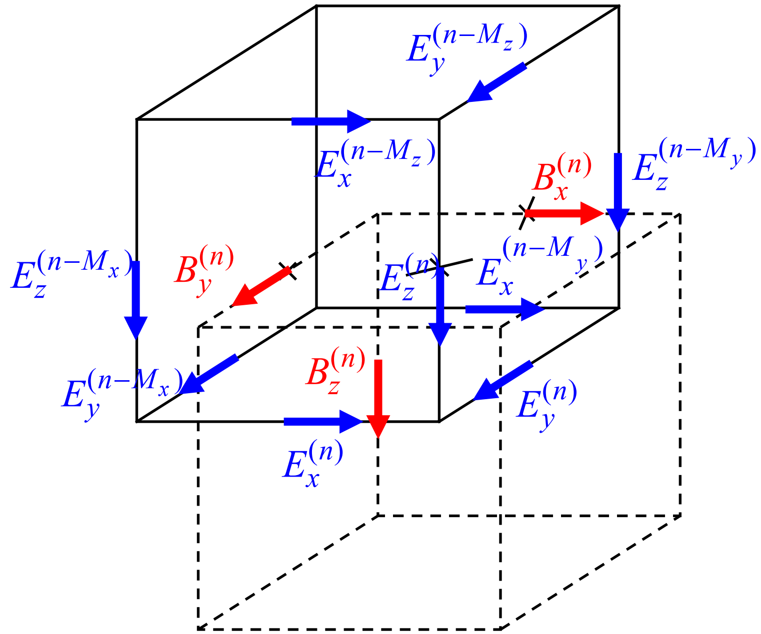
$$\frac{\mathrm{d}}{\mathrm{d}t}B_y^{(m)}(t)\Delta y\Delta z = -\left[-E_x^{(u)}(t)\Delta x + E_z^{(l)}(t)\Delta z + E_x^{(d)}(t)\Delta x - E_z^{(r)}(t)\Delta z\right] - J_{\mathrm{m}y}^{(m)}(t)\Delta y\Delta z$$

$$\frac{\mathrm{d}}{\mathrm{d}t}B_z^{(m)}(t)\Delta y\Delta z = -\left[E_y^{(u)}(t)\Delta y + E_z^{(f)}(t)\Delta z - E_y^{(d)}(t)\Delta y + E_z^{(b)}(t)\Delta z\right] - J_{\mathrm{m}z}^{(m)}(t)\Delta y\Delta z$$

# Dual-Orthogonal Grid System in Space / Dual-orthogonales Gittersystem im Raum

3-D / 3D




| | | |
|---|---|---|
| Primary grid / Primäres Gitter | $G \perp \tilde{G}$ | Secondary (dual) grid Sekundäres (duales) Gitter |
| Primary grid / Primäres Gitter | $G = M$ | Material grid Materialgitter |

Global node numbering / Globale Gitternummerierung

$$n = 1 + M_x(n_x - 1) + M_y(n_y - 1) + M_z(n_z - 1)$$

$$n = 1, 2, \ldots, N = N_x N_y N_z$$

$$M_x = 1$$

$$M_y = N_x$$

$$M_z = N_x N_y$$

# End of Lecture 8 /
# Ende der 8. Vorlesung