Radio Science, Volume 31, Number 6, Pages 1919–1930, November–December 1996

Electromagnetic and elastodynamic point source excitation of unbounded homogeneous anisotropic media

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Abstract. Plane electromagnetic as well as plane elastodynamic waves in anisotropic media exhibit a different direction of their phase and energy propagation, resulting in slowness and group velocity surfaces. Of course, the availability of plane wave solutions gives rise to a spectral plane wave decomposition of point source excitations, i.e., Green's functions. Unfortunately, the coordinate-free closed-form solution of dyadic (electric) Green's functions in the **R**ω space is only known for electromagnetic (generalized) uniaxial media. Utilizing the relation between phase and group velocities of plane waves in uniaxial media we have been able to show that the phase and amplitude of the Green's function is related to the group velocity; i.e., time domain wave fronts reproduce group velocity surfaces. This has also been verified through numerical results obtained by the three-dimensional (3-D) electromagnetic finite integration technique code. In elastodynamics, where similar analytical results for anisotropic media are not available, we confirm this behavior with our numerical 3-D elastodynamic finite integration technique code. For electromagnetic uniaxial media, we present an analytic method to derive the dyadic far-field Green's function in **R**ω space from **K**ω space directly by utilizing the duality principle between wave vectors and ray vectors without performing the 3-D inverse Fourier transform from **K**ω space to **R**ω space analytically.

Introduction

Volume sources, or equivalent surface sources based on a mathematical formulation of Huygens' principle, which radiate into unbounded media require the knowledge of "free space" Green's functions for computational purposes. In particular, in nondestructive testing with ultrasound, anisotropic materials like fiber-reinforced composites or austenitic welds have recently found considerable practical interest [e.g., *Thompson and Chimenti*, 1996], but unfortunately, the knowledge concerning explicit mathematical expressions for appropriate Green's functions (tensors), even for the simplest case of transverse isotropy, is still very sparse. Therefore, it is not even possible to evaluate the radiation pattern of "ultrasonic antennas" for such materials, because the far-field directivity of Green's tensors would be needed.

In order to assess some conjectures concerning the mathematical far-field structure of elastodynamic Green's tensors for anisotropic media, we have a closer look at their electromagnetic counterparts. Especially for electromagnetically uniaxial anisotropic media, the Green's dyadic is explicitly known in the spatial domain, and a novel concept to evaluate its far field can therefore be analytically confirmed. For electromagnetically biaxial anisotropic and elastodynamically transverse isotropic media we propose the same procedure, especially since numerical results confirm some of our conjectures.

General treatments of electromagnetic waves in anisotropic media can be found in various textbooks [e.g., *Felsen and Marcuvitz*, 1973; *Chen*, 1983; *Lindell*, 1992]. A coordinate-free closed-form solution of the "free-space" dyadic Green's function has been given by *Chen* [1983] for uniaxial homogeneous media and by *Weiglhofer* [1990] for general uniaxial media. Several analytic methods and dyadic Green's functions have been collected by *Weiglhofer* [1993]. Recently, properties of the dyadic Green's function for biaxial media have been discussed by *Cottis and Kondylis* [1995], and Green's dyadics for a special class of bianisotropic media have been presented by *Lindell and Olyslager* [1995]. The dyadic Green's function and radiation in uniaxially anisotropic media, especially the asymptotic evaluation of the integrals using a method developed by *Lighthill* [1960], and the far-field of an electric dipole were studied by *Chen*

Paper number 96RS02155.
0048-6604/96/96RS-02155$11.00

[1973]. The dipole radiation into a homogeneous anisotropic medium was studied by *Mittra and Deschamps* [1963] using the pole extraction method in the spectral domain.

In elastodynamics [e.g., *Ben-Menahem and Singh*, 1981; *van der Hijden*, 1987; *Tverdokhlebov and Rose*, 1988], similar results for the dyadic and triadic Green's functions in **R**ω space are unknown. Only for transversely isotropic media the dyadic and triadic Green;s functions in **K**ω space have been given by *Spies* [1994a].

The basic ideas of this paper are enumerated as follows:

1. Suppose, an explicit representation of a Green's tensor is known in a spatial spectral domain given by the three-dimensional (3-D) Fourier vector **K** (**K** space). In any case, this representation is structured as follows: A dyadic **K** function operates on the Fourier spectrum of a scalar Green's function, which is singular on the phase velocity surface, the Ewald surface, which is the Ewald sphere for isotropic media.
2. The 3-D Fourier inverse of the scalar Green's function exhibits wave fronts (Huygens-type wavelets) in the space time domain, whose spatial geometry is identical to the group or energy velocity surface (wave surface).
3. In space time domain the dyadic **K** prefactor is a dyadic differential operator, whose result for the far field would be needed to yield the directivity pattern of Green's dyadics.
4. In isotropic media with the wave number k, the stationary phase evaluation of the Fourier inverse yields $\mathbf{K} = k\hat{\mathbf{R}}$ with $\hat{\mathbf{R}}$ being a unit vector into observation direction. This means that in a particular $\hat{\mathbf{R}}$ direction, only those spectral components that reside in the neighbourhood of that direction in **K** space contribute significantly. For anisotropic materials the stationary phase evaluation is generally by no means straightforward, but experience with the isotropic case suggests that selection of a certain energy propagation direction (ray vector direction) requires **K** space integration around the pertinent wave vector direction, which is given by the phase velocity surface. Since rays and wave vectors are not parallel, their respective angle has to be evaluated, and this can be achieved applying the duality principle.

The above steps will be analytically confirmed for the electromagnetic uniaxial case. Throughout the paper all material parameters are assumed constant in the frequency regime. Vectors, dyadics, and tetradics appear boldface, sans serif, and sans serif with underline. A coordinate-free notation is used.

Green's Dyadics in Electromagnetics

For unbounded homogeneous, lossless, and nonmagnetic anisotropic media characterized by a relative permittivity tensor $\boldsymbol{\varepsilon}_r$ with $\mu_r = 1$, if a time dependence $\sim \exp(-j\omega t)$ is assumed, the "generalized" Maxwell equations are [e.g., *Chen*, 1983]

$$j\omega\mu_0\mathbf{H}(\mathbf{R}, \omega) = \nabla \times \mathbf{E}(\mathbf{R}, \omega) + \mathbf{J}_m(\mathbf{R}, \omega), \tag{1}$$

$$j\omega\varepsilon_0\boldsymbol{\varepsilon}_r \cdot \mathbf{E}(\mathbf{R}, \omega) = -\nabla \times \mathbf{H}(\mathbf{R}, \omega) + \mathbf{J}_e(\mathbf{R}, \omega), \tag{2}$$

with the constitutive relations

$$\mathbf{B}(\mathbf{R}, \omega) = \mu_0\mathbf{H}(\mathbf{R}, \omega), \tag{3}$$

$$\mathbf{D}(\mathbf{R}, \omega) = \varepsilon_0\boldsymbol{\varepsilon}_r \cdot \mathbf{E}(\mathbf{R}, \omega). \tag{4}$$

The vector fields **E**, **H**, **D**, **B**, $\mathbf{J}_m$, and $\mathbf{J}_e$ stand for electric field, magnetic field, electric flux density, magnetic flux density, magnetic current density, and electric current density. **R** is the position vector, and ω is the circular frequency; μ_0 and ε_0 denote the permeability and permittivity of free space resulting in the phase velocity $c_0 = 1/\sqrt{\mu_0\varepsilon_0}$.

A volume source for an electric dipole is defined by the electric current density

$$\mathbf{J}_e(\mathbf{R}, \omega) = -j\omega p(\omega)\delta(\mathbf{R})\hat{\mathbf{p}}, \tag{5}$$

where $p(\omega)$ is the Fourier spectrum, $\delta(\mathbf{R})$ is the 3-D Dirac delta function, and $\hat{\mathbf{p}}$ is a unit vector. Then the radiated electric field is given by

$$\mathbf{E}(\mathbf{R}, \omega) = j\omega\mu_0 \int_{V'} \mathsf{G}_e(\mathbf{R} - \mathbf{R}', \omega) \cdot \mathbf{J}_e(\mathbf{R}', \omega)\, d^3\mathbf{R}', \tag{6}$$

$$\mathbf{E}(\mathbf{R}, \omega) = \omega^2\mu_0 p(\omega)\mathsf{G}_e(\mathbf{R}, \omega) \cdot \hat{\mathbf{p}}, \tag{7}$$

with the dyadic (electric) Green's function G_e. The far field of the electric field is then

$$\mathbf{E}^{\text{far}}(\mathbf{R}, \omega) = \omega^2\mu_0 p(\omega)\mathsf{G}_e^{\text{far}}(\mathbf{R}, \omega) \cdot \hat{\mathbf{p}}. \tag{8}$$

Homogeneous Isotropic Media

For homogeneous isotropic media with $\boldsymbol{\varepsilon}_r = \varepsilon_r\mathsf{I}$ the dyadic (electric) Green's function $\mathsf{G}_e^{\text{iso}}$ satisfies the differential equation

$$[\nabla\nabla - (\Delta - k_0^2\varepsilon_r)\mathsf{I}] \cdot \mathsf{G}_e^{\text{iso}}(\mathbf{R} - \mathbf{R}', \omega) = \mathsf{I}\delta(\mathbf{R} - \mathbf{R}'), \tag{9}$$

where Δ is $\nabla \cdot \nabla$, I is the unit dyadic or idemfactor defined by $\mathsf{I} = \delta_{ij}\mathbf{e}_i\mathbf{e}_j$ utilizing the summation convention with δ_{ij} as the Kronecker symbol, and $k_0 = \omega/c_0$ is the wave number of free space. A 3-D Fourier transform of Green's dyadic according to

$$\widetilde{\mathsf{G}}_e(\mathbf{K}, \omega) = \int_{-\infty}^{+\infty} \mathsf{G}_e(\mathbf{R}, \omega) \exp(-j\mathbf{K} \cdot \mathbf{R})\, d^3\mathbf{K} \tag{10}$$

yields

$$\underbrace{[(K^2 - k_0^2\varepsilon_r)\mathsf{I} - \mathbf{KK}]}_{\widetilde{\mathsf{W}}_e^{\mathrm{iso}}(\mathbf{K}, \omega)} \cdot \widetilde{\mathsf{G}}_e^{\mathrm{iso}}(\mathbf{K}, \omega) = \mathsf{I} \tag{11}$$

or

$$\widetilde{\mathsf{G}}_e^{\mathrm{iso}}(\mathbf{K}, \omega) = [\widetilde{\mathsf{W}}_e^{\mathrm{iso}}(\mathbf{K}, \omega)]^{-1} \cdot \mathsf{I}, \tag{12}$$

where $\widetilde{\mathsf{W}}_e^{\mathrm{iso}}(\mathbf{K}, \omega)$ is the so-called wave dyadic. (From $\det\{\widetilde{\mathsf{W}}_e(\mathbf{K}, \omega)\} = 0$ we can derive wave vectors $\mathbf{K}_\eta(\hat{\mathbf{K}})$, wave numbers $k_\eta(\hat{\mathbf{K}})$, slownesses $s_\eta(\hat{\mathbf{K}})$, phase velocities $c_{\mathrm{ph},\eta}(\hat{\mathbf{K}})$, group velocity vectors $\mathbf{c}_{\mathrm{gr},\eta}(\hat{\mathbf{K}})$, energy velocity vectors $\mathbf{c}_{\mathrm{E},\eta}(\hat{\mathbf{K}})$, and ray vectors $\mathbf{l}_\eta(\hat{\mathbf{K}}) = \mathbf{c}_{\mathrm{gr},\eta}(\hat{\mathbf{K}})/\omega$ for each wave mode η; isotropic media, one ordinary mode o; uniaxial media, one ordinary mode o and one extraordinary mode e; and biaxial media, two extraordinary modes $e1$ and $e2$.) For isotropic media the inversion of the wave dyadic can be performed analytically, giving

$$\widetilde{\mathsf{G}}_e^{\mathrm{iso}}(\mathbf{K}, \omega) = [\widetilde{\mathsf{W}}_e^{\mathrm{iso}^{-1}}(\mathbf{K}, \omega)]^{-1}, \tag{13}$$

$$\widetilde{\mathsf{G}}_e^{\mathrm{iso}}(\mathbf{K}, \omega) = \left(\mathsf{I} - \frac{1}{k_0^2\varepsilon_r}\mathbf{KK}\right)\frac{1}{K^2 - k_0^2\varepsilon_r}, \tag{14}$$

$$\widetilde{\mathsf{G}}_e^{\mathrm{iso}}(\mathbf{K}, \omega) = \widetilde{\mathsf{D}}_e^{\mathrm{iso}}(\mathbf{K}, \omega)\widetilde{G}_e^{\mathrm{iso}}(\mathbf{K}, \omega). \tag{15}$$

After an explicit 3-D inverse Fourier transform we obtain the $\mathbf{R}\omega$ representation

$$\mathsf{G}_e^{\mathrm{iso}}(\mathbf{R}, \omega) = \left(\mathsf{I} + \frac{1}{k_0^2\varepsilon_r}\nabla\nabla\right)\frac{\exp(j\omega\sqrt{\varepsilon_r}R/c_0)}{4\pi R}, \tag{16}$$

$$\mathsf{G}_e^{\mathrm{iso}}(\mathbf{R}, \omega) = \mathsf{D}_e^{\mathrm{iso}}(\nabla, \omega)G_e^{\mathrm{iso}}(\mathbf{R}, \omega). \tag{17}$$

For isotropic media, the far-field representation of the dyadic Green's function can be evaluated in the following two ways:

1. Evaluate $\nabla\nabla$ in (16) for $k_0R \gg 1$ and retain only $1/R$ terms. We obtain therefore

$$\nabla \overset{\mathrm{far}}{\Rightarrow} jk_0\sqrt{\varepsilon_r}\hat{\mathbf{R}}. \tag{18}$$

2. Perform a 3-D Fourier inversion of $\widetilde{\mathsf{G}}_e^{\mathrm{iso}}(\mathbf{K}, \omega)$ in (14) with stationary phase arguments, which yields

$$\mathbf{K} \overset{\mathrm{far}}{\Rightarrow} k_0\sqrt{\varepsilon_r}\hat{\mathbf{R}}. \tag{19}$$

For (16) we obtain with (18) the far-field representation in $\mathbf{R}\omega$ space

$$\mathsf{G}_e^{\mathrm{iso,far}}(\mathbf{R}, \omega) = (\mathsf{I} - \hat{\mathbf{R}}\hat{\mathbf{R}})\frac{\exp(j\omega\sqrt{\varepsilon_r}R/c_0)}{4\pi R}, \tag{20}$$

$$\mathsf{G}_e^{\mathrm{iso,far}}(\mathbf{R}, \omega) = \mathsf{D}_e^{\mathrm{iso,far}}(\hat{\mathbf{R}})G_e^{\mathrm{iso,far}}(\mathbf{R}, \omega), \tag{21}$$

and in the time domain ($\mathbf{R}t$ space)

$$\mathsf{G}_e^{\mathrm{iso,far}}(\mathbf{R}, t) = (\mathsf{I} - \hat{\mathbf{R}}\hat{\mathbf{R}})\frac{\delta(t - \sqrt{\varepsilon_r}R/c_0)}{4\pi R}, \tag{22}$$

$$\mathsf{G}_e^{\mathrm{iso,far}}(\mathbf{R}, t) = \mathsf{D}_e^{\mathrm{iso,far}}(\hat{\mathbf{R}})G_e^{\mathrm{iso,far}}(\mathbf{R}, t), \tag{23}$$

with $\delta(x)$ being the 1-D Dirac delta function. Equations (14) and (22) show explicitly that phase surfaces in $\mathbf{K}\omega$ space (Ewald spheres) are wave fronts in $\mathbf{R}t$ space.

Now the idea, based on (19) and the knowledge of the scalar Green's function, is to get the far field without 3-D Fourier inversion of the wave dyadic $\mathsf{W}(\mathbf{K}, \omega)$.

Homogeneous Uniaxial Anisotropic Media

For uniaxial media the differential equation for the dyadic Green's function reads,

$$(\nabla\nabla - \Delta\mathsf{I} - k_0^2\boldsymbol{\varepsilon}_r^{\mathrm{uni}}) \cdot \mathsf{G}_e^{\mathrm{uni}}(\mathbf{R} - \mathbf{R}'; \omega) = \mathsf{I}\delta(\mathbf{R} - \mathbf{R}'), \tag{24}$$

with the relative permittivity tensor

$$\boldsymbol{\varepsilon}_r^{\mathrm{uni}} = \varepsilon_\perp\mathsf{I} + (\varepsilon_\parallel - \varepsilon_\perp)\hat{\mathbf{c}}\hat{\mathbf{c}}, \tag{25}$$

where $\hat{\mathbf{c}}$ is the optical axis that is an eigenvector of $\boldsymbol{\varepsilon}_r^{\mathrm{uni}}$ corresponding to the eigenvalue $\varepsilon_\parallel$, and $\varepsilon_\perp$ is the other eigenvalue of $\boldsymbol{\varepsilon}_r^{\mathrm{uni}}$. The wave dyadic has the form

$$\widetilde{\mathsf{W}}_e^{\mathrm{uni}}(\mathbf{K}, \omega) = (\mathbf{K} \cdot \mathbf{K} - k_0^2\varepsilon_\perp)\mathsf{I} - \mathbf{KK} + (\varepsilon_\perp - \varepsilon_\parallel)k_0^2\hat{\mathbf{c}}\hat{\mathbf{c}}. \tag{26}$$

The electromagnetic wave field in uniaxial media separates into an ordinary (o) and an extraordinary (e) wave mode. For each mode we evaluate analytically from the dispersion equation

$$\det\{\mathsf{W}_{\mathrm{e}}^{\mathrm{uni}}(\mathbf{K},\omega)\} = k_0^2(k_0^2\varepsilon_\perp - \mathbf{K}\cdot\mathbf{K}) \cdot (k_0^2\varepsilon_\perp\varepsilon_\parallel - \mathbf{K}\cdot\boldsymbol{\varepsilon}_{\mathrm{r}}^{\mathrm{uni}}\cdot\mathbf{K}) = 0 \quad (27)$$

the wave numbers

$$K_o = k_0\sqrt{\varepsilon_\perp}, \quad (28)$$

$$K_e(\hat{\mathbf{K}}) = k_0 \frac{\sqrt{\varepsilon_\perp\varepsilon_\parallel}}{\sqrt{\hat{\mathbf{K}}\cdot\boldsymbol{\varepsilon}_{\mathrm{r}}^{\mathrm{uni}}\cdot\hat{\mathbf{K}}}}, \quad (29)$$

group velocity vectors

$$\mathbf{c}_{\mathrm{gr},o}(\hat{\mathbf{K}}) = \frac{c_0}{\sqrt{\varepsilon_\perp}}\hat{\mathbf{K}}, \quad (30)$$

$$\mathbf{c}_{\mathrm{gr},e}(\hat{\mathbf{K}}) = \frac{c_0^2}{\omega\sqrt{\varepsilon_\perp\varepsilon_\parallel}}\boldsymbol{\varepsilon}_{\mathrm{r}}^{\mathrm{uni}}\cdot\hat{\mathbf{K}}, \quad (31)$$

which equal the energy velocity vectors, and ray vectors

$$\mathbf{l}_o(\hat{\mathbf{K}}) = \frac{c_0}{\omega\sqrt{\varepsilon_\perp}}\hat{\mathbf{K}}, \quad (32)$$

$$\mathbf{l}_e(\hat{\mathbf{K}}) = \frac{c_0^2}{\omega^2\sqrt{\varepsilon_\perp\varepsilon_\parallel}}\boldsymbol{\varepsilon}_{\mathrm{r}}^{\mathrm{uni}}\cdot\hat{\mathbf{K}}. \quad (33)$$

The above expressions for the extraordinary mode exhibit explicitly the dependence on the permittivity tensor and the propagation direction.

Determination of the inverse of the wave dyadic in (26) yields the dyadic (electric) Green's function for uniaxial media in $\mathbf{K}\omega$ space; it reads [*Chen*, 1983],

$$\tilde{\mathsf{G}}_{\mathrm{e}}^{\mathrm{uni}}(\mathbf{K},\omega) = \frac{(\mathbf{K}\times\hat{\mathbf{c}})(\mathbf{K}\times\hat{\mathbf{c}})}{(\mathbf{K}\times\hat{\mathbf{c}})^2}\frac{1}{K^2 - k_0^2\varepsilon_\perp} + \left[\varepsilon_\perp\varepsilon_\parallel(\boldsymbol{\varepsilon}_{\mathrm{r}}^{\mathrm{uni}})^{-1} - \frac{1}{k_0^2}\mathbf{K}\mathbf{K} - \varepsilon_\parallel\frac{(\mathbf{K}\times\hat{\mathbf{c}})(\mathbf{K}\times\hat{\mathbf{c}})}{(\mathbf{K}\times\hat{\mathbf{c}})^2}\right] \cdot \frac{1}{\mathbf{K}\cdot\boldsymbol{\varepsilon}_{\mathrm{r}}^{\mathrm{uni}}\cdot\mathbf{K} - k_0^2\varepsilon_\perp\varepsilon_\parallel}, \quad (34)$$

$$\tilde{\mathsf{G}}_{\mathrm{e}}^{\mathrm{uni}}(\mathbf{K},\omega) = \tilde{\mathsf{D}}_{\mathrm{e},o}^{\mathrm{uni}}(\mathbf{K},\omega)\tilde{G}_{\mathrm{e},o}^{\mathrm{uni}}(\mathbf{K},\omega) + \tilde{\mathsf{D}}_{\mathrm{e},e}^{\mathrm{uni}}(\mathbf{K},\omega)\tilde{G}_{\mathrm{e},e}^{\mathrm{uni}}(\mathbf{K},\omega), \quad (35)$$

with the dyadic prefactors and the scalar Green's functions for the ordinary mode $\tilde{\mathsf{D}}_{\mathrm{e},o}^{\mathrm{uni}}(\mathbf{K},\omega)$, $\tilde{G}_{\mathrm{e},o}^{\mathrm{uni}}(\mathbf{K},\omega)$ and extraordinary mode $\tilde{\mathsf{D}}_{\mathrm{e},e}^{\mathrm{uni}}(\mathbf{K},\omega)$, $\tilde{G}_{\mathrm{e},e}^{\mathrm{uni}}(\mathbf{K},\omega)$. The explicit 3-D inverse Fourier transform of (34) with respect to $\mathbf{K}$ has been given by *Chen* [1983]. The obtained dyadic Green's function in $\mathbf{R}\omega$ space reads for the ordinary mode,

$$\mathsf{G}_{\mathrm{e},o}^{\mathrm{uni}}(\mathbf{R},\omega) = \frac{\sqrt{\varepsilon_\perp}}{c_0} \cdot \left\{\frac{(\hat{\mathbf{R}}\times\hat{\mathbf{c}})(\hat{\mathbf{R}}\times\hat{\mathbf{c}})}{(\hat{\mathbf{R}}\times\hat{\mathbf{c}})^2} + \frac{1}{jk_0R}\frac{1}{\varepsilon_\perp}\frac{c_0}{c_{\mathrm{gr},o}}\frac{1}{(\hat{\mathbf{R}}\times\hat{\mathbf{c}})^2} \cdot \left[\mathsf{I} - \hat{\mathbf{c}}\hat{\mathbf{c}} - 2\frac{(\hat{\mathbf{R}}\times\hat{\mathbf{c}})(\hat{\mathbf{R}}\times\hat{\mathbf{c}})}{(\hat{\mathbf{R}}\times\hat{\mathbf{c}})^2}\right]\right\}\frac{\exp[j\omega R/c_{\mathrm{gr},o}]}{4\pi R/c_{\mathrm{gr},o}}, \quad (36)$$

$$\mathsf{G}_{\mathrm{e},o}^{\mathrm{uni}}(\mathbf{R},\omega) = \mathsf{D}_{\mathrm{e},o}^{\mathrm{uni}}(\mathbf{R},\omega)G_{\mathrm{e},o}^{\mathrm{uni}}(\mathbf{R},\omega), \quad (37)$$

and for the extraordinary mode

$$\begin{aligned}\mathsf{G}_{\mathrm{e},e}^{\mathrm{uni}}(\mathbf{R},\omega) = {}& \frac{1}{c_0}\frac{\varepsilon_\parallel}{\sqrt{\varepsilon_\perp}} \cdot \left\{\varepsilon_\perp(\boldsymbol{\varepsilon}_{\mathrm{r}}^{\mathrm{uni}})^{-1} - \frac{c_{\mathrm{gr},e}^2(\hat{\mathbf{R}})}{c_0^2}\varepsilon_\parallel\varepsilon_\perp^2(\boldsymbol{\varepsilon}_{\mathrm{r}}^{\mathrm{uni}})^{-1}\cdot\hat{\mathbf{R}}\hat{\mathbf{R}}\cdot(\boldsymbol{\varepsilon}_{\mathrm{r}}^{\mathrm{uni}})^{-1}\right.\\ & - \frac{(\hat{\mathbf{R}}\times\hat{\mathbf{c}})(\hat{\mathbf{R}}\times\hat{\mathbf{c}})}{(\hat{\mathbf{R}}\times\hat{\mathbf{c}})^2} + \left[\frac{1}{jk_0R}\frac{c_{\mathrm{gr},e}(\hat{\mathbf{R}})}{c_0} + \frac{1}{k_0^2R^2}\frac{c_{\mathrm{gr},e}^2(\hat{\mathbf{R}})}{c_0^2}\right]\\ & \cdot 3\frac{c_{\mathrm{gr},e}^2(\hat{\mathbf{R}})}{c_0^2}\varepsilon_\parallel\varepsilon_\perp^2(\boldsymbol{\varepsilon}_{\mathrm{r}}^{\mathrm{uni}})^{-1}\cdot\hat{\mathbf{R}}\hat{\mathbf{R}}\cdot(\boldsymbol{\varepsilon}_{\mathrm{r}}^{\mathrm{uni}})^{-1}\\ & + \left[\frac{1}{jk_0R}\frac{c_{\mathrm{gr},e}(\hat{\mathbf{R}})}{c_0} + \frac{1}{k_0^2R^2}\frac{c_{\mathrm{gr},e}^2(\hat{\mathbf{R}})}{c_0^2}\right]\varepsilon_\perp(\boldsymbol{\varepsilon}_{\mathrm{r}}^{\mathrm{uni}})^{-1}\\ & - \frac{1}{jk_0R}\frac{1}{\varepsilon_\parallel}\frac{c_0}{c_{\mathrm{gr},e}(\hat{\mathbf{R}})}\frac{1}{(\hat{\mathbf{R}}\times\hat{\mathbf{c}})^2}\\ & \left.\cdot\left[\mathsf{I} - \hat{\mathbf{c}}\hat{\mathbf{c}} - 2\frac{(\hat{\mathbf{R}}\times\hat{\mathbf{c}})(\hat{\mathbf{R}}\times\hat{\mathbf{c}})}{(\hat{\mathbf{R}}\times\hat{\mathbf{c}})^2}\right]\right\}\frac{\exp[j\omega R/c_{\mathrm{gr},e}(\hat{\mathbf{R}})]}{4\pi R/c_{\mathrm{gr},e}(\hat{\mathbf{R}})},\end{aligned} \quad (38)$$

$$\mathsf{G}_{\mathrm{e},e}^{\mathrm{uni}}(\mathbf{R},\omega) = \mathsf{D}_{\mathrm{e},e}^{\mathrm{uni}}(\mathbf{R},\omega)G_{\mathrm{e},e}^{\mathrm{uni}}(\mathbf{R},\omega), \quad (39)$$

with the group velocities (see (53) and (55))

$$c_{\mathrm{gr},o} = \frac{c_0}{\sqrt{\varepsilon_\perp}}, \quad (40)$$

$$c_{\mathrm{gr},e}(\hat{\mathbf{R}}) = \frac{c_0}{\sqrt{\varepsilon_\perp\varepsilon_\parallel}}\frac{1}{\sqrt{\hat{\mathbf{R}}\cdot(\boldsymbol{\varepsilon}_{\mathrm{r}}^{\mathrm{uni}})^{-1}\cdot\hat{\mathbf{R}}}}. \quad (41)$$

This clearly shows that the phase and amplitude of the scalar Green's functions $G_{\mathrm{e},o}^{\mathrm{uni}}(\mathbf{R},\omega)$ and $G_{\mathrm{e},e}^{\mathrm{uni}}(\mathbf{R},\omega)$ is

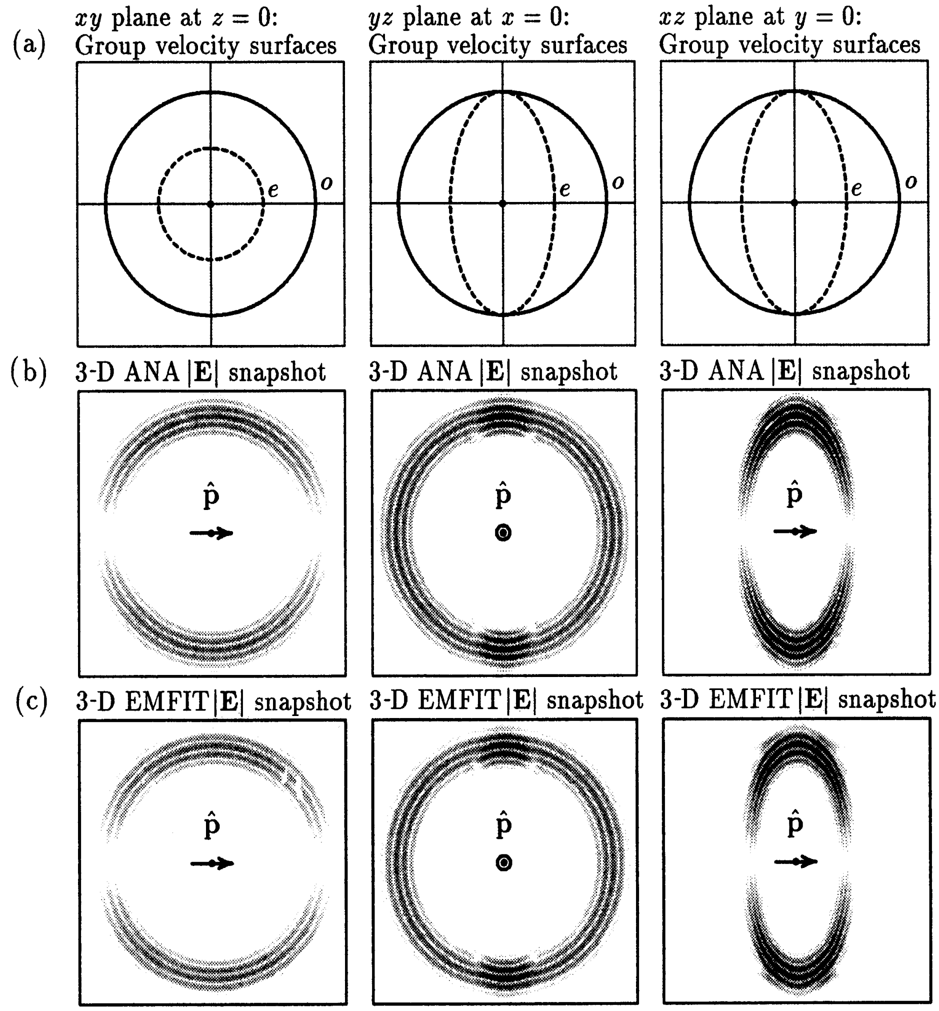

Figure 1. Positive ($\varepsilon_\parallel > \varepsilon_\perp$) uniaxial medium with $\hat{\mathbf{c}} = \mathbf{e}_z$ and $\varepsilon_\parallel = 4\varepsilon_\perp$. (a) Group velocity surfaces of the ordinary mode o and extraordinary mode e. (b) Three-dimensional ANA $|\mathbf{E}|$ snapshots of wave fronts at time point $t = t_1$ radiated by an electric dipole given by $\mathbf{J}_e(\mathbf{R}, t) = f_{\mathrm{RC2}}(t)\delta(\mathbf{R})\hat{\mathbf{p}}$ with $\hat{\mathbf{p}} = \mathbf{e}_x$ and a center frequency of $f_c = 10$ GHz. (c) Three-dimensional EMFIT $|\mathbf{E}|$ snapshots of wave fronts at $t = t_1 = 200\Delta t$ of the same source. EMFIT parameters are spatial domain size, 40 cm × 40 cm × 40 cm; uniform grid with mesh width $\Delta x = 2$ mm; total grid size, 200^3 voxels = 201^3 nodes; time step width; $\Delta t = 0.5\Delta x/c_0 = 3.33$ ps. The applied 3-D EMFIT code is of second order in time and space.

related to the group velocities in (40) and (41). This has been called spatial scaling by *Felsen and Marcuvitz* [1973], and it can be independently derived with the Cagniard–de Hoop method [e.g., *de Hoop*, 1960; *van der Hijden*, 1987] even for the biaxial case.

We exploited Chen's formulas to compute wave-front snapshots of $\mathbf{E}$ in the time domain; therefore we use the short-term 3-D analytical (ANA). Figure 1 displays group velocity surfaces related to computed wave fronts of the 3-D ANA $|\mathbf{E}|$ snapshots according to Chen's formulas and the 3-D EMFIT $|\mathbf{E}|$ snapshots of the 3-D electromagnetic finite integration technique (EMFIT) code [*Marklein*, 1994] for a transient electric dipole excitation, i.e., $|\mathbf{E}| = \sqrt{E_x^2 + E_y^2 + E_z^2}$. The time history of the transient electric dipole is prescribed by a raised cosine function with two cycles, called RC2 pulse, defined by

$$f_{\mathrm{RC2}}(t) = \begin{cases} [1 - \cos(\pi f_c t)]\cos(2\pi f_c t) & 0 < t < T \\ 0 & \text{otherwise'} \end{cases} \tag{42}$$

with the center frequency f_c and the pulse length $T = 2/f_c$ (see Figure 2). The EMFIT code is a 3-D time domain modeling code based on Maxwell's equations

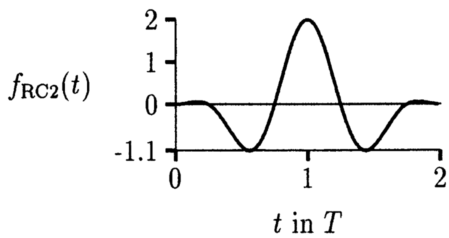

Figure 2. Time history of the *RC*2 pulse.

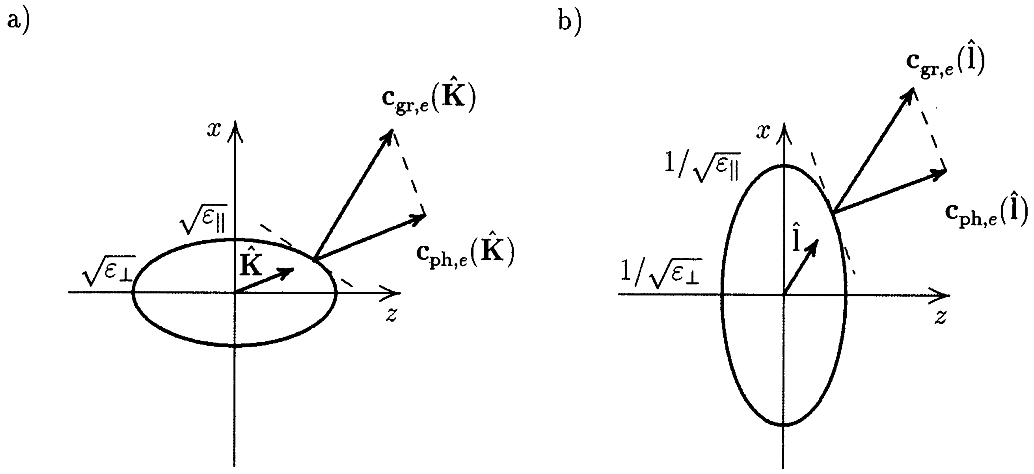

Figure 3. The duality principle illustrated for the extraordinary mode of a negative uniaxial medium: (a) the wave vector surface normalized to k_0 and (b) the ray vector surface normalized to $1/k_0$.

in $\mathbf{R}t$ space in integral form. The EMFIT algorithm compares to the solution of Maxwell's equations by the finite integration algorithm (MAFIA) [*MAFIA Collaboration*, 1991; *Bartsch et al.*, 1992], which in the time domain is essentially the well-known finite difference time domain method [*Taflove*, 1995]. Figure 1 confirms for the uniaxial case that plane wave group velocity surfaces represent wave fronts in the time domain.

Application of the Duality Principle

In order to calculate radiation patterns of sources radiating in uniaxial media independently from *Chen*'s [1983] formulas (equations (36)–(39)), we take advantage of the duality between wave vector surfaces and ray vector surfaces, called the duality principle (DP) [*Chen*, 1983]. Figure 3 shows this duality for the extraordinary mode. For the plane wave ansatz according to

$$\mathbf{E}(\mathbf{R}, \omega) = \mathbf{E}_0(\omega) \exp(j\mathbf{K} \cdot \mathbf{R}), \tag{43}$$

$$\mathbf{H}(\mathbf{R}, \omega) = \mathbf{H}_0(\omega) \exp(j\mathbf{K} \cdot \mathbf{R}), \tag{44}$$

we write Maxwell's equations as a function of $\mathbf{K}$ and $\mathbf{l}$ separately, which defines the following two sets of equations:

Set 1

$$\mathbf{D}_0(\omega) = -\frac{1}{\omega}\mathbf{K} \times \mathbf{H}_0(\omega), \tag{45a}$$

$$\mathbf{B}_0(\omega) = \frac{1}{\omega}\mathbf{K} \times \mathbf{E}_0(\omega), \tag{45b}$$

$$\mathbf{D}_0(\omega) = \varepsilon_0 \boldsymbol{\varepsilon}_{\mathrm{r}} \cdot \mathbf{E}_0(\omega), \tag{45c}$$

$$\mathbf{B}_0(\omega) = \mu_0 \mathbf{H}_0(\omega), \tag{45d}$$

$$\frac{\mathbf{K}}{\omega} = \frac{\mathbf{D}_0(\omega) \times \mathbf{B}_0(\omega)}{\mathbf{E}_0(\omega) \cdot \mathbf{D}_0(\omega)}, \tag{45e}$$

Set 2

$$\mathbf{E}_0(\omega) = -\omega \mathbf{l} \times \mathbf{B}_0(\omega), \tag{45f}$$

$$\mathbf{H}_0(\omega) = \omega \mathbf{l} \times \mathbf{D}_0(\omega), \tag{45g}$$

$$\mathbf{E}_0(\omega) = (\varepsilon_0 \boldsymbol{\varepsilon}_{\mathrm{r}})^{-1} \cdot \mathbf{D}_0(\omega), \tag{45h}$$

$$\mathbf{H}_0(\omega) = \mu_0^{-1} \mathbf{B}_0(\omega), \tag{45i}$$

$$\omega \mathbf{l} = \frac{\mathbf{E}_0(\omega) \times \mathbf{H}_0(\omega)}{\mathbf{E}_0(\omega) \cdot \mathbf{D}_0(\omega)}. \tag{45j}$$

Either set can be obtained from the other by interchanging the symbols in the following way:

$$\mathbf{D}_0(\omega) \Leftrightarrow \mathbf{E}_0(\omega), \tag{46}$$

$$\mathbf{B}_0(\omega) \Leftrightarrow \mathbf{H}_0(\omega), \tag{47}$$

$$\frac{1}{\omega}\mathbf{K} \Leftrightarrow \omega \mathbf{l}, \tag{48}$$

$$\varepsilon_0 \boldsymbol{\varepsilon}_{\mathrm{r}} \Leftrightarrow (\varepsilon_0 \boldsymbol{\varepsilon}_{\mathrm{r}})^{-1}, \tag{49}$$

$$\mu_0 \Leftrightarrow 1/\mu_0. \tag{50}$$

Relations (49) and (50) define also

$$c_0 \Leftrightarrow 1/c_0. \tag{51}$$

At first we use the duality principle to derive the amplitudes of the ray and group velocity vectors from (28) and (29): We derive for the ordinary mode

$$l_o = \omega\, c_0/\sqrt{\varepsilon_\perp}, \tag{52}$$

$$c_{\mathrm{gr},o} = c_0/\sqrt{\varepsilon_\perp}, \tag{53}$$

and for the extraordinary mode

$$l_e(\hat{\mathbf{l}}_e) = \omega\,\frac{c_0}{\sqrt{\varepsilon_\perp\varepsilon_\|}}\,\frac{1}{\sqrt{\hat{\mathbf{l}}_e\cdot(\boldsymbol{\varepsilon}_\mathrm{r}^{\mathrm{uni}})^{-1}\cdot\hat{\mathbf{l}}_e}}, \tag{54}$$

$$c_{\mathrm{gr},e}(\hat{\mathbf{l}}_e) = \frac{c_0}{\sqrt{\varepsilon_\perp\varepsilon_\|}}\,\frac{1}{\sqrt{\hat{\mathbf{l}}_e\cdot(\boldsymbol{\varepsilon}_\mathrm{r}^{\mathrm{uni}})^{-1}\cdot\hat{\mathbf{l}}_e}}. \tag{55}$$

To evaluate far-field relations for the two modes (like (19)), we write for each mode the $\mathbf{K}_\eta$ as a function of the ray vector $\mathbf{l}_\eta$ and identify (or replace) the unit ray vector $\hat{\mathbf{l}}_\eta$ with the observation vector $\hat{\mathbf{R}}$.

For the ordinary mode we apply DP to (33) and get the **K** vector

$$\mathbf{K}_o(\hat{\mathbf{l}}_o) = k_0^2\varepsilon_\perp\mathbf{l}_o(\hat{\mathbf{l}}_o), \tag{56}$$

or with (52)

$$\mathbf{K}_o(\hat{\mathbf{l}}_o) = k_0\varepsilon_\perp\hat{\mathbf{l}}_o. \tag{57}$$

Now we identify the unit ray vector $\hat{\mathbf{l}}_o$ with observation direction $\hat{\mathbf{R}}$ and find the far-field approximation for the **K** vector of the ordinary mode

$$\mathbf{K}_o \overset{\mathrm{far}}{\Rightarrow} k_0\sqrt{\varepsilon_\perp}\,\hat{\mathbf{R}}. \tag{58}$$

The **K** vector of the extraordinary mode is given from (33) via DP, i.e.,

$$\mathbf{K}_e(\hat{\mathbf{l}}) = \frac{\omega^2}{c_0^2}\,\varepsilon_\perp\varepsilon_\|(\boldsymbol{\varepsilon}_\mathrm{r}^{\mathrm{uni}})^{-1}\cdot\mathbf{l}_e(\hat{\mathbf{l}}_e). \tag{59}$$

With $c_{\mathrm{gr}} = l/\omega$ and (55), the **K** vector has the form

$$\mathbf{K}_e(\hat{\mathbf{l}}) = \frac{\omega}{c_0}\,\varepsilon_\perp\varepsilon_\|\,\frac{c_{\mathrm{gr},e}(\hat{\mathbf{l}}_e)}{c_0}\,(\boldsymbol{\varepsilon}_\mathrm{r}^{\mathrm{uni}})^{-1}\cdot\hat{\mathbf{l}}_e. \tag{60}$$

Identification of the unit ray vector $\hat{\mathbf{l}}_e$ with the observation direction $\hat{\mathbf{R}}$ yields the pendant to (58) for the extraordinary mode,

$$\mathbf{K}_e \overset{\mathrm{far}}{\Rightarrow} k_0\varepsilon_\|\varepsilon_\perp\,\frac{c_{\mathrm{gr},e}(\hat{\mathbf{R}})}{c_0}\,(\boldsymbol{\varepsilon}_\mathrm{r}^{\mathrm{uni}})^{-1}\cdot\hat{\mathbf{R}}. \tag{61}$$

Putting (58) and (61) into (34) yields the dyadic (electric) far-field Green's function (different from that used by *Chen* [1983, p. 382, equation (10.82)] for the electric far field of an electric dipole) directly from **K**ω space,

$$\mathsf{G}_{eo}^{\mathrm{uni,far}}(\mathbf{R},\omega) = \frac{\sqrt{\varepsilon_\perp}}{c_0}\,\frac{(\hat{\mathbf{R}}\times\hat{\mathbf{c}})(\hat{\mathbf{R}}\times\hat{\mathbf{c}})}{(\hat{\mathbf{R}}\times\hat{\mathbf{c}})^2}\,\frac{\exp[j\omega R/c_{\mathrm{gr},o}(\hat{\mathbf{R}})]}{4\pi R/c_{\mathrm{gr},o}(\hat{\mathbf{R}})}, \tag{62}$$

$$\mathsf{G}_{eo}^{\mathrm{uni,far}}(\mathbf{R},\omega) = \mathsf{D}_{\mathrm{e},o}^{\mathrm{uni,far}}(\hat{\mathbf{R}})G_{\mathrm{e},o}^{\mathrm{uni,far}}(\mathbf{R},\omega), \tag{63}$$

$$\mathsf{G}_{ee}^{\mathrm{uni,far}}(\mathbf{R},\omega) = \frac{1}{c_0}\frac{\varepsilon_\|}{\sqrt{\varepsilon_\perp}}\Bigg\{\varepsilon_\perp(\boldsymbol{\varepsilon}_\mathrm{r}^{\mathrm{uni}})^{-1} - \frac{[c_{\mathrm{gr},e}(\hat{\mathbf{R}})]^2}{c_0^2}\,\varepsilon_\|\varepsilon_\perp^2(\boldsymbol{\varepsilon}_\mathrm{r}^{\mathrm{uni}})^{-1}\cdot\hat{\mathbf{R}}\hat{\mathbf{R}}\cdot(\boldsymbol{\varepsilon}_\mathrm{r}^{\mathrm{uni}})^{-1} - \frac{(\hat{\mathbf{R}}\times\hat{\mathbf{c}})(\hat{\mathbf{R}}\times\hat{\mathbf{c}})}{(\hat{\mathbf{R}}\times\hat{\mathbf{c}})^2}\Bigg\}\,\frac{\exp[j\omega R/c_{\mathrm{gr},e}(\hat{\mathbf{R}})]}{4\pi R/c_{\mathrm{gr},e}(\hat{\mathbf{R}})} \tag{64}$$

$$\mathsf{G}_{ee}^{\mathrm{uni,far}}(\mathbf{R},\omega) = \mathsf{D}_{\mathrm{e},e}^{\mathrm{uni,far}}(\hat{\mathbf{R}})G_{\mathrm{e},e}^{\mathrm{uni,far}}(\mathbf{R},\omega), \tag{65}$$

with the group velocities $c_{\mathrm{gr},o}$ and $c_{\mathrm{gr},e}(\hat{\mathbf{R}})$ defined in (40) and (41).

Of course, the far-field representations (62)–(65) can be also derived from Chen's formulas (36)–(39) for $k_0R \gg 1$.

Homogeneous Biaxial Media

For biaxial media the differential equation for Green's dyadic reads,

$$(\nabla\nabla - \Delta\mathsf{I} - k_0^2\boldsymbol{\varepsilon}_\mathrm{r}^{\mathrm{bi}})\cdot\mathsf{G}_\mathrm{e}^{\mathrm{bi}}(\mathbf{R}-\mathbf{R}',\omega) = \mathsf{I}\delta(\mathbf{R}-\mathbf{R}') \tag{66}$$

with the relative permittivity tensor

$$\boldsymbol{\varepsilon}_\mathrm{r}^{\mathrm{bi}} = \varepsilon_1\mathsf{I} + (\varepsilon_3-\varepsilon_2)\,\mathrm{sym}\,\{\hat{\mathbf{c}}_1\hat{\mathbf{c}}_2\} \tag{67}$$

and the two optical axes $\hat{\mathbf{c}}_1$ and $\hat{\mathbf{c}}_2$. The permittivity constants ε_i are given according to $\boldsymbol{\varepsilon}_\mathrm{r}^{\mathrm{bi}} = \varepsilon_i\mathbf{e}_i\mathbf{e}_i$. The operator "sym$\{\hat{\mathbf{c}}_1\hat{\mathbf{c}}_2\}$" denotes the symmetric part of the dyadic $\hat{\mathbf{c}}_1\hat{\mathbf{c}}_2$. In general, the symmetric and antisymmetric part of a dyadic (e.g., $\mathsf{A} = \mathbf{ab}$) reads, $\mathsf{A} = \mathrm{sym}\,\{\mathsf{A}\} + \mathrm{asym}\,\{\mathsf{A}\} = \mathrm{sym}\,\{\mathbf{ab}\} + \mathrm{asym}\,\{\mathbf{ab}\}$ with $\mathrm{sym}\,\{\mathsf{A}\} = \mathrm{sym}\,\{\mathbf{ab}\} = \frac{1}{2}(\mathbf{ab}+\mathbf{ba})$ and $\mathrm{asym}\,\{\mathsf{A}\} = \mathrm{asym}\,\{\mathbf{ab}\} = \frac{1}{2}(\mathbf{ab}-\mathbf{ba})$.

The wave dyadic for biaxial media is given by

$$\mathsf{W}_\mathrm{e}^{\mathrm{bi}}(\mathbf{K},\omega) = (\mathbf{K}\cdot\mathbf{K} - k_0^2\varepsilon_1)\mathsf{I} - \mathbf{KK} + (\varepsilon_2-\varepsilon_3)k_0^2\,\mathrm{sym}\,\{\hat{\mathbf{c}}_1\hat{\mathbf{c}}_2\}. \tag{68}$$

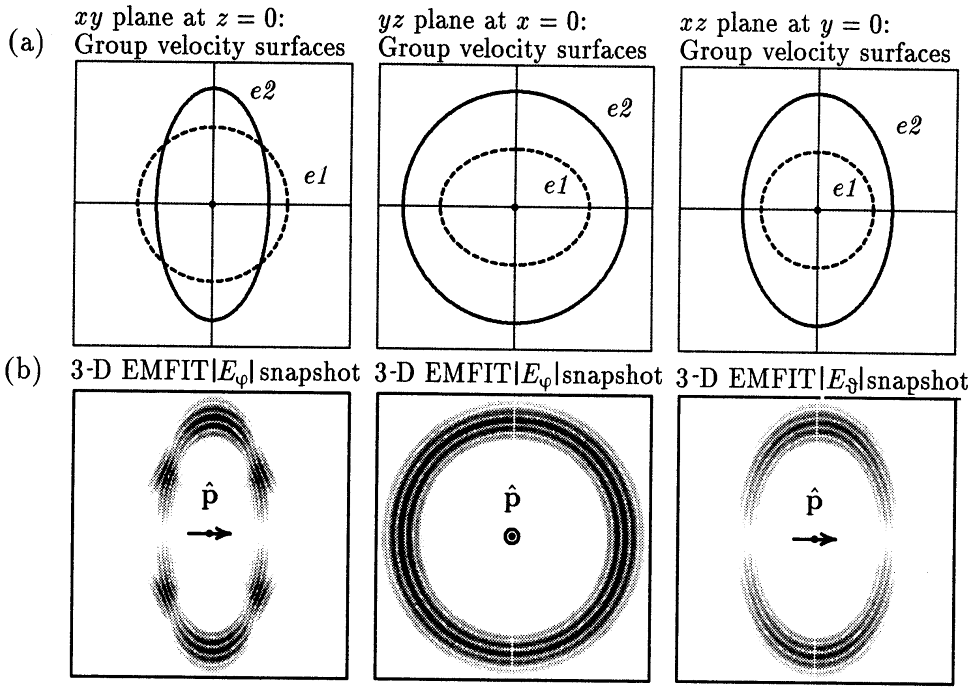

Figure 4. A biaxial medium with $\varepsilon_{r,xx} = \varepsilon_0$, $\varepsilon_{r,yy} = 4\varepsilon_0$, $\varepsilon_{r,zz} = 2.25\varepsilon_0$, and $\varepsilon_{r,ij} = 0$ for $i \neq j$. (a) Group velocity surfaces of the two extraordinary modes $e1$ and $e2$. (b) Three-dimensional EMFIT snapshots at time point $t = t_1 = 200\Delta t$ of E_φ in the xy plane at $z = 0$ and in the yz plane at $x = 0$, and of E_ϑ in the xz plane at $y = 0$ radiated by an electric dipole given by $\mathbf{J}_e(\mathbf{R}, t) = f_{RC2}(t)\delta(\mathbf{R})\hat{\mathbf{p}}$ with $\hat{\mathbf{p}} = \mathbf{e}_x$ and a center frequency of $f_c = 10$ GHz. EMFIT parameters are spatial domain size, 40 cm × 40 cm × 40 cm; uniform grid with mesh width $\Delta x = 2$ mm; total grid size, 200^3 voxels $= 201^3$ nodes; time step width, $\Delta t = 0.5\Delta x/c_0 = 3.33$ ps. The applied 3-D EMFIT code is of second order in time and space.

For the biaxial case, we only confirm numerically the fact that plane wave group velocity surfaces represent wave fronts in the time domain. Figure 4 shows the coincidence of the analytically obtained group velocity diagram with the time domain snapshots of the electric field computed by the 3-D EMFIT code. Here we calculated especially the nonzero spherical components E_R, E_ϑ, and E_φ via coordinate transform of the Cartesian components E_x, E_y, and E_z obtained by the 3-D EMFIT code.

Green's Dyadics in Elastodynamics

For unbounded homogeneous, lossless anisotropic media with the mass density at rest ρ_0 and the compliance tensor $\underline{\mathsf{s}}$ the governing equations of linear elastodynamics, Cauchy's equation of motion and the equation of deformation rate [e.g., *van der Hijden*, 1987], read with time dependence $\sim \exp(-j\omega t)$

$$-j\omega\rho_0\mathbf{v}(\mathbf{R}, \omega) = \nabla \cdot \mathsf{T}(\mathbf{R}, \omega) + \mathbf{f}(\mathbf{R}, \omega), \tag{69}$$

$$-j\omega\underline{\mathsf{s}} : \mathsf{T}(\mathbf{R}, \omega) = \mathrm{sym}\,\{\nabla\mathbf{v}(\mathbf{R}, \omega)\} + \mathsf{h}(\mathbf{R}, \omega), \tag{70}$$

with the constitutive relations

$$\mathbf{p}(\mathbf{R}, \omega) = \rho_0\mathbf{v}(\mathbf{R}, \omega), \tag{71}$$

$$\mathsf{S}(\mathbf{R}, \omega) = \underline{\mathsf{s}} : \mathsf{T}(\mathbf{R}, \omega). \tag{72}$$

The vector and tensor fields $\mathbf{v}$, T, $\mathbf{f}$, h, $\mathbf{p}$, and S are particle velocity vector, stress tensor (stress dyadic), volume force density vector, source of deformation rate dyadic, momentum density vector, and deformation tensor (deformation dyadic). The colon denotes the double-scalar product with the property $\mathbf{ab}:\mathbf{cd} = (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$. (This notation was introduced by *Ben-Menahem and Singh* [1981], and differs from that used in electromagnetics by such investigators as *Gibbs* [1913], *Lindell* [1992], and *Weiglhofer* [1993].)

The volume source density of a point force is defined by [*Ben-Menahem and Singh*, 1981]

$$\mathbf{f}(\mathbf{R}, \omega) = f(\omega)\delta(\mathbf{R})\hat{\mathbf{f}}, \tag{73}$$

where $f(\omega)$ is the Fourier spectrum and $\hat{\mathbf{f}}$ is a unit vector. The radiated particle displacement field ($\mathbf{u}(\mathbf{R}, \omega) = -\mathbf{v}(\mathbf{R}, \omega)/j\omega$) of the point force is represented by

$$\mathbf{u}(\mathbf{R}, \omega) = \int_{V'} \mathsf{G}_u(\mathbf{R} - \mathbf{R}', \omega) \cdot \mathbf{f}(\mathbf{R}', \omega)d^3\mathbf{R}', \tag{74}$$

$$\mathbf{u}(\mathbf{R}, \omega) = f(\omega)\mathsf{G}_{\mathrm{u}}(\mathbf{R}, \omega) \cdot \hat{\mathbf{f}}, \tag{75}$$

where G_{u} is the dyadic (particle displacement) Green's function. The far field of the radiated particle displacement is determined by

$$\mathbf{u}^{\mathrm{iso,far}}(\mathbf{R}, \omega) = f(\omega)\mathsf{G}_{\mathrm{u}}^{\mathrm{iso,far}}(\mathbf{R}, \omega) \cdot \hat{\mathbf{f}}, \tag{76}$$

which can be used to evaluate radiation pattern of the point force. The knowledge of the far-field Green's function is mandatory for the evaluation of the radiation pattern of an "ultrasonic antenna" [e.g., *Spiess*, 1994b].

Homogeneous Isotropic Media

For homogeneous isotropic media the differential equation for the dyadic Green's functions is

$$(\nabla \cdot \underline{\mathsf{c}}^{\mathrm{iso}} \cdot \nabla + \omega^2\rho_0\mathsf{I}) \cdot \mathsf{G}_{\mathrm{u}}(\mathbf{R} - \mathbf{R}', \omega) = -\mathsf{I}\delta(\mathbf{R} - \mathbf{R}') \tag{77}$$

with the stiffness tensor

$$\underline{\mathsf{c}}^{\mathrm{iso}} = \lambda\mathsf{II} + \mu(\mathsf{II}^{1324} + \mathsf{II}^{1342}), \tag{78}$$

which is the inverse of the compliance tensor $\mathsf{s}^{\mathrm{iso}}$, where the upper indical notation indicates transposition of elements, e.g., $\mathsf{II}^{1342} = \{\delta_{ij}\delta_{kl}\mathbf{e}_i\mathbf{e}_j\mathbf{e}_k\mathbf{e}_l\}^{1342} = \delta_{ij}\delta_{kl}\mathbf{e}_i\mathbf{e}_k\mathbf{e}_l\mathbf{e}_j = \delta_{il}\delta_{jk}\mathbf{e}_i\mathbf{e}_j\mathbf{e}_k\mathbf{e}_l$ with summation convention understood, and λ and μ are Lamé's parameters. The elastodynamic wave dyadic becomes the form

$$\widetilde{\mathsf{W}}_{\mathrm{u}}^{\mathrm{iso}}(\mathbf{K}, \omega) = -(\lambda + \mu)\mathbf{KK} + (-\mu\mathbf{K} \cdot \mathbf{K} + \rho_0\omega^2)\mathsf{I}. \tag{79}$$

The inversion of this wave dyadic [e.g., *Fellinger*, 1991] yields the elastodynamic dyadic (particle displacement) Green's function in $\mathbf{K}\omega$ space

$$\widetilde{\mathsf{G}}_{\mathrm{u}}^{\mathrm{iso}}(\mathbf{K}, \omega) = \frac{1}{\rho_0\omega^2}\mathbf{KK}\frac{1}{K^2 - k_p^2} + \frac{1}{\rho_0\omega^2}(k_s^2\mathsf{I} - \mathbf{KK})\frac{1}{K^2 - k_s^2}, \tag{80}$$

$$\widetilde{\mathsf{G}}_{\mathrm{u}}^{\mathrm{iso}}(\mathbf{K}, \omega) = \widetilde{\mathsf{D}}_{\mathrm{u},p}^{\mathrm{iso}}(\mathbf{K}, \omega)\widetilde{G}_{\mathrm{u},p}^{\mathrm{iso}}(\mathbf{K}, \omega) + \widetilde{\mathsf{D}}_{\mathrm{u},s}(\mathbf{K}, \omega)\widetilde{G}_{\mathrm{u},s}^{\mathrm{iso}}(\mathbf{K}, \omega), \tag{81}$$

where $k_p = \omega/c_{\mathrm{ph},p}$ and $k_s = \omega/c_{\mathrm{ph},s}$ are the wave numbers with the phase velocities $c_{\mathrm{ph},p} = \sqrt{(\lambda + 2\mu)/\rho_0}$ and $c_{\mathrm{ph},s} = \sqrt{\mu/\rho_0}$ of the pressure (p) and shear wave (s) for isotropic media. In $\mathbf{R}\omega$ space the dyadic Green's function takes the form

$$\mathsf{G}_{\mathrm{u}}^{\mathrm{iso}}(\mathbf{R}, \omega) = -\frac{1}{\rho_0\omega^2}\nabla\nabla\frac{\exp(j\omega R/c_{\mathrm{ph},p})}{4\pi R} + \frac{1}{\rho_0\omega^2}(k_s^2\mathsf{I} + \nabla\nabla)\frac{\exp(j\omega R/c_{\mathrm{ph},s})}{4\pi R} \tag{82}$$

$$\mathsf{G}_{\mathrm{u}}^{\mathrm{iso}}(\mathbf{R}, \omega) = \mathsf{D}_{\mathrm{u},p}^{\mathrm{iso}}(\nabla, \omega)G_{\mathrm{u},p}^{\mathrm{iso}}(\mathbf{R}, \omega) + \mathsf{D}_{\mathrm{u},s}^{\mathrm{iso}}(\nabla, \omega)G_{\mathrm{u},s}^{\mathrm{iso}}(\mathbf{R}, \omega). \tag{83}$$

Introducing the far-field approximation $\mathbf{K}_\eta \overset{\mathrm{far}}{\Rightarrow} k_\eta\hat{\mathbf{R}}$, $\eta = p, s$ in (80) or $\nabla \overset{\mathrm{far}}{\Rightarrow} jk_\eta\hat{\mathbf{R}}$, $\eta = p, s$ in (82), we obtain the dyadic far-field Green's function

$$\mathsf{G}_{\mathrm{u}}^{\mathrm{iso,far}}(\mathbf{R}, \omega) = \frac{\hat{\mathbf{R}}\hat{\mathbf{R}}}{\rho_0 c_{\mathrm{ph},p}^2}\frac{\exp(j\omega R/c_{\mathrm{ph},p})}{4\pi R} + \frac{\mathsf{I} + \hat{\mathbf{R}}\hat{\mathbf{R}}}{\rho_0 c_{\mathrm{ph},s}^2}\frac{\exp(j\omega R/c_{\mathrm{ph},s})}{4\pi R} \tag{84}$$

$$\mathsf{G}_{\mathrm{u}}^{\mathrm{iso,far}} = \mathsf{D}_{\mathrm{u},p}^{\mathrm{iso,far}}(\hat{\mathbf{R}})G_{\mathrm{u},p}^{\mathrm{iso,far}}(\mathbf{R}, \omega) + \mathsf{D}_{\mathrm{u},s}^{\mathrm{iso,far}}(\hat{\mathbf{R}})G_{\mathrm{u},s}^{\mathrm{iso,far}}(\mathbf{R}, \omega). \tag{85}$$

The far-field representation (84) can then be used for the development of an elastodynamic Kirchhoff-type inverse scattering algorithm based on Huygens integrals [*Langenberg et al.*, 1993]. For the formulation of the Huygens integral a triadic Green's function must be derived from the dyadic Green's function [e.g., *Fellinger*, 1991].

Homogeneous Transversely Isotropic Media

In elastodynamics, except for isotropic media, no closed-form analytical solutions for dyadic and triadic Green's functions in $\mathbf{R}\omega$ space are available. Nevertheless, for example, transversely isotropic media like austenitic steel or unidirectional fiber-reinforced composites are of practical interest today. Because of the complex physics of elastodynamic waves in such media, there is a definite need for analytic solutions in order to get a better understanding from a theoretical point of view. The stiffness tensor for transversely isotropic media, it is the same as for hexagonal crystals, reads in coordinate-free form [e.g., *Fellinger et al.*, 1995],

$$\begin{aligned}\underline{\mathsf{c}}^{\mathrm{ti}} = {} & (c_2 - 2c_5)\mathsf{II} + c_5(\mathsf{II}^{1324} + \mathsf{II}^{1342}) \\ & + [c_1 + c_2 - 2(c_3 + 2c_4)]\hat{\mathbf{m}}\hat{\mathbf{m}}\hat{\mathbf{m}}\hat{\mathbf{m}} \\ & + (c_3 - c_2 + 2c_5)(\mathsf{I}\hat{\mathbf{m}}\hat{\mathbf{m}} + \hat{\mathbf{m}}\hat{\mathbf{m}}\mathsf{I}) + (c_4 - c_5) \\ & \cdot (\mathsf{I}\hat{\mathbf{m}}\hat{\mathbf{m}}^{1324} + \mathsf{I}\hat{\mathbf{m}}\hat{\mathbf{m}}^{1342} + \hat{\mathbf{m}}\hat{\mathbf{m}}\mathsf{I}^{1324} + \hat{\mathbf{m}}\hat{\mathbf{m}}\mathsf{I}^{1342}),\end{aligned} \tag{86}$$

where c_i $(i = 1, \cdots, 5)$ are the five elastic constants and $\hat{\mathbf{m}}$ denotes the unit vector perpendicular to which isotropy holds, which refers to the "fiber direction." For transversely isotropic media the wave dyad and the dyadic and triadic Green's functions in $\mathbf{K}\omega$ space have been derived by *Spies* [1994a]. The wave dyad reads, for transversely isotropic media,

$$\widetilde{\mathsf{W}}_{\mathrm{u}}^{\mathrm{ti}}(\mathbf{K}, \omega) = \alpha \mathsf{I} + \beta \mathbf{K}\mathbf{K} + \gamma \hat{\mathbf{m}}\hat{\mathbf{m}} + 2\epsilon \operatorname{sym}\{\mathbf{K}\hat{\mathbf{m}}\} \tag{87}$$

with

$$\alpha = c_5 K^2 + (c_4 - c_5)(\hat{\mathbf{m}} \cdot \mathbf{K})^2 - \rho_0 \omega^2, \tag{88}$$

$$\beta = c_2 - c_5, \tag{89}$$

$$\gamma = [c_1 + c_2 - 2(c_3 + 2c_4)](\hat{\mathbf{m}} \cdot \mathbf{K})^2 + (c_4 - c_5)K^2, \tag{90}$$

$$\epsilon = [c_3 + c_4 - (c_2 - c_5)](\hat{\mathbf{m}} \cdot \mathbf{K}). \tag{91}$$

Then the dyadic Green's function is given by

$$\widetilde{\mathsf{G}}_{\mathrm{u}}^{\mathrm{ti}}(\mathbf{K}, \omega) = \sum_{\eta} \widetilde{\mathsf{G}}_{\mathrm{u},\eta}^{\mathrm{ti}}(\mathbf{K}, \omega), \qquad \eta = qP, qSV, SH \tag{92}$$

$$\widetilde{\mathsf{G}}_{\mathrm{u}}^{\mathrm{ti}}(\mathbf{K}, \omega) = \sum_{\eta} \widetilde{\mathsf{D}}_{\mathrm{u},\eta}^{\mathrm{ti}}(\mathbf{K}, \omega) \widetilde{G}_{\mathrm{u},\eta}^{\mathrm{ti}}(\mathbf{K}, \omega), \tag{93}$$

$$\eta = qP, qSV, SH$$

with

$$\widetilde{\mathsf{D}}_{\mathrm{u},\eta}^{\mathrm{ti}}(\mathbf{K}, \omega) = \frac{\delta_{\eta SH}}{c_5} \mathsf{I} + \frac{\beta \mathbf{K}\mathbf{K} + \gamma \hat{\mathbf{m}}\hat{\mathbf{m}} + 2\epsilon \operatorname{sym}\{\mathbf{K}\hat{\mathbf{m}}\}}{c_2 c_4}$$
$$\times \sum_{\kappa \neq \eta} \frac{(1 - \delta_{\eta SH})(1 - \delta_{\kappa SH})}{k_\kappa^2 - k_\eta^2}$$
$$- \frac{(\beta\gamma - \epsilon^2)[\mathbf{K}\mathbf{K} + K^2 \hat{\mathbf{m}}\hat{\mathbf{m}} - 2(\hat{\mathbf{m}} \cdot \mathbf{K}) \operatorname{sym}\{\mathbf{K}\hat{\mathbf{m}}\}]}{c_2 c_4 c_5}$$
$$\cdot \prod_{\kappa \neq \eta} \frac{1}{K_\kappa^2 - K_\eta^2}, \qquad \eta, \kappa = qP, qSV, SH, \tag{94}$$

$$\widetilde{G}_{\mathrm{u},\eta}^{\mathrm{ti}}(\mathbf{K}, \omega) = \frac{1}{K^2 - K_\eta^2}. \tag{95}$$

$\widetilde{\mathsf{G}}_{\mathrm{u}}^{\mathrm{ti}}(\mathbf{K}, \omega)$ decomposes into a quasi-pressure qP, a quasi-shear vertical qSV, and a pure shear horizontal wave SH (for an arbitrary anisotropic medium it separates into three quasi-waves called quasi-pressure qP, quasi-shear first $qS1$, and quasi-shear second $qS2$ [*van der Hijden*, 1987]). The determination of the 3-D inverse Fourier transform of $\widetilde{\mathsf{G}}_{\mathrm{u}}^{\mathrm{ti}}(\mathbf{K}, \omega)$ is not possible, since the resulting Fourier integrals cannot be solved analytically. Up to now, no analytical closed-form solutions of the dyadic and triadic Green's functions in $\mathbf{R}\omega$ space are available. However, for transversely isotropic media explicit analytic expressions for wave numbers, slownesses, group velocity vectors, and ray vectors for the three modes can be evaluated [e.g., *Spies*, 1994a; *Fellinger et al.*, 1995]. For example, for the transversely isotropic medium austenitic steel 308SS, Figure 5 compares group velocity surfaces of the three modes with wave front snapshots of $|\mathbf{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}$ calculated with the 3-D elastodynamic finite integration technique (EFIT) code [*Fellinger et al.*, 1995; *Marklein et al.*, 1996] for a transient point force excitation with an RC2 time history. The EFIT code is the elastodynamic pendant of the EMFIT code. The point force and fiber direction are horizontally oriented, i.e., $\hat{\mathbf{f}} = \hat{\mathbf{m}} = \mathbf{e}_x$. Because $\hat{\mathbf{f}} = \hat{\mathbf{m}}$, the SH mode is not excited. Again, the group velocity surfaces coincide with the EFIT wave fronts. This reveals that also in elastodynamics the phase and amplitude of the scalar Green's function $G_{\mathrm{u},\eta}^{\mathrm{ti}}(\mathbf{R}, t)$ for each mode η is defined by the pertinent group velocity $c_{\mathrm{gr},\eta}^{\mathrm{ti}}$.

Conclusion

For electromagnetics and elastodynamics we have illustrated via comparison of analytic and numerical results that the group velocity diagrams are representing wave fronts in the time domain. We have shown for electromagnetic uniaxial media that the dyadic Green's function always separates into a dyadic prefactor and a scalar Green's function, and the phase and amplitude of the scalar Green's function is determined by the group velocity. Then we have shown that one can derive the far field in $\mathbf{R}\omega$ space from $\mathbf{K}\omega$ space directly if the far-field identification for the wave vector $\mathbf{K}$ for each mode is known. This has been demonstrated analytically for the electromagnetic uniaxial case using the duality principle.

In addition, according to the procedure in the electromagnetic uniaxial and biaxial case, we make the following conjecture for elastodynamic anisotropic media: if we know the dyadic Green's function in $\mathbf{K}\omega$ space and the far-field identification for the wave vector $\mathbf{K}_\eta$ for each mode η as a result of the duality principle between the wave vector $\mathbf{K}$ and ray vector $\mathbf{l}$, then the dyadic far-field Green's function can be

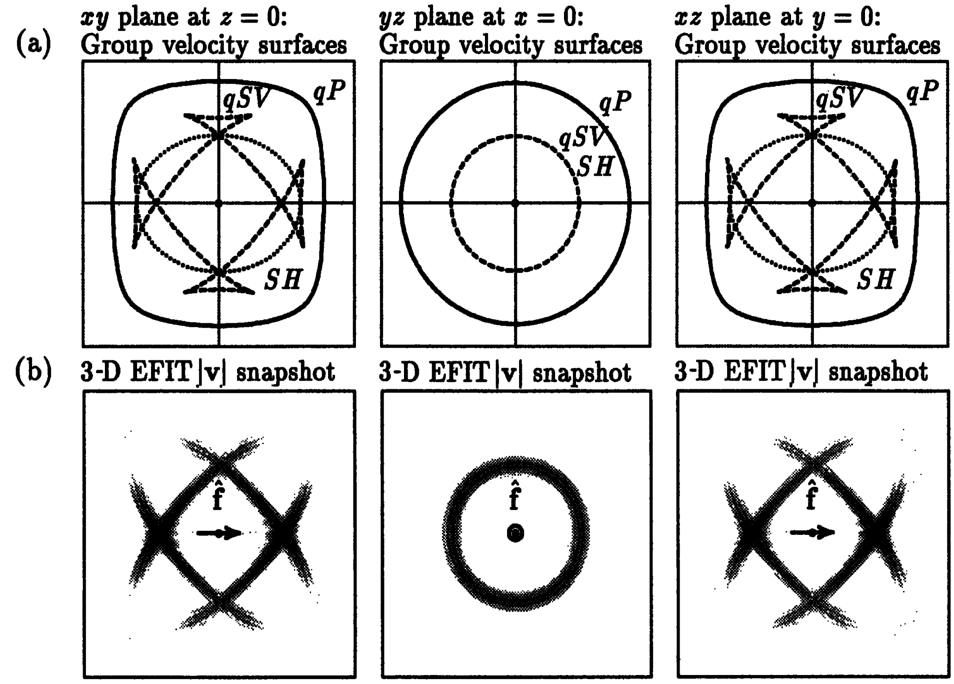

Figure 5. Transversely isotropic medium austenitic steel 308SS with $\hat{\mathbf{m}} = \mathbf{e}_x$ and the material parameters $c_1 = 216$ GPa, $c_2 = 262.75$ GPa, $c_3 = 145$ GPa, $c_4 = 129$ GPa, $c_5 = 82.5$ GPa, and $\rho_0 = 7800$ kg/m^3. (a) Group velocity surfaces of the three modes qP, qSV, and SH. (b) Three-dimensional EFIT $|\mathbf{v}|$ snapshots of wave fronts at time point $t = t_1 = 230\Delta t$ radiated by a point force given by $\mathbf{f}(\mathbf{R}, t) = f_{\mathrm{RC2}}(t)\delta(\mathbf{R})\hat{\mathbf{f}}$ with $\hat{\mathbf{f}} = \mathbf{e}_x$ and a center frequency of $f_c = 4$ MHz. EFIT parameters are spatial domain size, 2 cm × 2 cm × 2 cm; uniform grid with mesh width $\Delta x = 100$ μm; total grid size, 200^3 voxels = 201^3 nodes; time step width, $\Delta t = 0.45\Delta x/5850.71$ s/m = 7.69 ns. The applied 3-D EFIT code is of second order in time and fourth order in space.

derived from $\mathbf{K}\omega$ space without carrying out the 3-D inverse Fourier transform $\mathbf{G}(\mathbf{R}, \omega)$ ●—○ $\widetilde{\mathbf{W}}^{-1}(\mathbf{K}, \omega)$ analytically.

References

Bartsch, et al., Solution of Maxwell's equations, *Comput. Phys. Commun.*, *72*, 22–39, 199?.

Ben-Menahem, A., and S. J. Singh, *Seismic Waves and Sources*, Springer-Verlag, New York, 1981.

Chen, H. C., Dyadic Green's function and radiation in a uniaxially anisotropic medium, *Int. J. Electron.*, *35*, 623–640, 1973.

Chen, H. C., *Theory of Electromagnetic Waves*, McGraw-Hill, New York, 1983.

Cottis, P. G., and G. D. Kondylis, Properties of the dyadic Green's function for an unbounded anisotropic medium, *IEEE Trans. Antennas Propag.*, *43*, 154–161, 1995.

de Hoop, A. T., A modification of Cagniard's method for solving seismic pulse problems, *Appl. Sci. Res.*, *Sect. B*, *8*, 349–356, 1960.

Fellinger, P., A technique for numerical solution of elastic wave propagation problems in time-domain by direct discretization of the governing equations of elastodynamics, (in German), Ph.D. thesis, Univ. of Kassel, Kassel, Germany, 1991.

Fellinger, P., R. Marklein, K. J. Langenberg, and S. Klaholz, Numerical modeling of elastic wave propagation and scattering with EFIT, *Wave Motion*, *21*, 47–66, 1995.

Felsen, L. B., and N. Marcuvitz, *Radiation and Scattering of Waves*, Prentice-Hall, Englewood Cliffs, N. J., 1973.

Gibbs, J. W., *Vector Analysis*, Yale University Press, New Haven, Conn., 1913.

Langenberg, K. J., P. Fellinger, R. Marklein, P. Zanger, K. Mayer, and T. Kreutter, Inverse methods and imaging, in *Evaluation of Materials and Structures by Quantitative Ultrasonics*, edited by J. D. Achenbach, pp. 317–398, Springer-Verlag, New York, 1993.

Lighthill, M. J., Studies on magneto-hydrodynamic waves and other anisotropic wave motions, *Philos. Trans. R. Soc. London, Ser. A*, *252*, 397, 1960.

Lindell, I. V., *Methods for Electromagnetic Field Analysis*, Clarendon Press, Oxford, 1992.

Lindell, I. V., and F. Olyslager, Duality transformations, Green dyadics and plane-wave solutions for a class of bianisotropic media, *J. Electromagn. Waves Appl.*, *9*, 85–96, 1995.

MAFIA Collaboration, *MAFIA User Guide, Release 3.1*, CST G.m.b.H., Darmstadt, Germany, 1991.

Marklein, R., Numerical modeling of electromagnetic waves in anisotropic media, report, Univ. of Kassel, Kassel, Germany, 1994.

Marklein, R., K. J. Langenberg, S. Klaholz, and J. Kostka, Ultrasonic modeling of real-life NDT situations: Applications and further developments, in *Review of Progress in Quantitative Nondestructive Evaluation*, vol. 15, edited by D. O. Thompson and E. E. Chimenti, Plenum, New York, 1996.

Mittra, R., and G. A. Deschamps, Field solution for a dipole in an anisotropic medium, in *Electromagnetic Theory and Antennas*, edited by E. C. Jordan, pp. 495–512, 1963.

Spies, M., Elastic wave propagation in general transversely isotropic media: Green's functions and elastodynamic holography, *J. Acoust. Soc. Am.*, *96*, 1144–1157, 1994a.

Spies, M., Transducer-modeling in general transversely isotropic media via point-source-synthesis: Theory, *J. Nondestructive Eval.*, *13*, 85–99, 1994b.

Taflove, A., *Computational Electrodynamics: The Finite-Difference Time-Domain Method*, Artech House, Norwood, Mass., 1995.

Thompson, D. O., and D. E. Chimenti (Eds.), *Review of Progress in Quantitative Nondestructive Evaluation*, vol. 15, Plenum, New York, 1996.

Tverdokhlebov, A., and J. L. Rose, On Green's function for elastic waves in anisotropic media, *J. Acoust. Soc. Am.*, *83*, 118–121, 1988.

van der Hijden, J. H. M. T., *Propagation of Transient Elastic Waves in Stratified Anisotropic Media*, North-Holland, New York, 1987.

Weiglhofer, W. S., Dyadic Green's function for general uniaxial media, *IEE Proc.*, *Part H*, *137*, 5–10, 1990.

Weiglhofer, W. S., Analytic methods and free-space dyadic Green's function, *Radio Sci.*, *28*, 847–857, 1993.

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(Received September 28, 1995; revised May 16, 1996; accepted July 12, 1996.)